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SUBJECTS FOR MATHEMATICAL ESSAYS

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SUBJECTS FOR MATHEMATICAL ESSAYS

BY

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BIRMINGHAM

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PREFACE.

THE object of what are here called "mathematical essays" is to co-ordinate a pupil's knowledge on certain subjects which are not specially dealt with in text-books. The essays, of which outlines are given in the first part of this book, are of the following types :

(i) A group of theorems on one subject, the theorems in ordinary text-books being often scattered in one or several volumes (*e.g.*, Essays 19, 26, 70, 91) ;

(ii) A series of questions leading up to the solution of an important problem (*e.g.*, Essays 79, 87, 90, 93) ;

(iii) A collection of different methods of proving the same theorem (*e.g.*, Essays 28, 61, 75, 78) ;

(iv) A series of applications of the same theorem (*e.g.*, Essays 37, 48, 75) ;

(v) A classification of tests of the same geometrical condition (*e.g.*, Essays 3, 4, 40, 41).

The subjects given in the first part (Essays 1-100) are, as a rule, of an elementary character. In several of these, a question, which throws light on different subjects, is repeated. Those given in the second part (Essays 101-200) are taken from papers set for entrance scholarships in the Trinity and Pembroke groups of Cambridge colleges from 1905 and 1907 respectively.

I should be grateful for notices of any errors that may be found in the text or answers, and for any suggestions from teachers for the improvement of the book.

CHARLES DAVISON.

BIRMINGHAM, *November*, 1914.

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PART I.

1.

TESTS OF DIVISIBILITY.

1. Show that a number is divisible by 9 if the sum of its digits be divisible by 9.

2. Show that a number is divisible by 11 if the difference between the sum of the digits in the odd places and the sum of the digits in the even places be either zero or divisible by 11.

3. Prove the following method of testing the accuracy of a multiplication example: Divide the sums of the digits in the multiplicand, multiplier and product by 9, then the product of the first two remainders when divided by 9 gives a remainder equal to the third remainder.

4. Arrange the digits 2, 3, 4, 5, 7 so as to form a number divisible by 132, and explain the reasons for the arrangement.

5. In a division sum, the divisor is 429, the four right-hand digits of the dividend are 2414, and the remainder is 132; find the three right-hand digits of the quotient.

6. When a certain number is multiplied by 13, the product consists entirely of nines; find the number.

7. The product of two numbers is $ab47c$, where a , b , c are integers less than 10, and one of the numbers is 792; find the other number.

8. The number $23ab7c$, in which a , b , c stand for obliterated digits, is known to be divisible by 792; find the missing digits.

M. E.

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2.

ELEMENTARY PROPERTIES OF NUMBERS.

1. The product of two numbers consisting of 3 and 4 digits must consist of 6 or 7 digits ; and the product of three numbers consisting of 3, 4 and 4 digits must consist of 9, 10 or 11 digits.

2. Show that, of any six consecutive numbers (except the first six), not more than two can be prime numbers.

3. Prove that the difference between the squares of any two odd numbers is always divisible by 8, and by 24 if neither of the numbers be divisible by 3.

4. If the sum of two square numbers be divisible by 3, show that this sum is also divisible by 9.

5. If a square number end in 6, the figure in the tens' place must be odd ; if it end in any other number, the figure in the tens' place must be even.

6. If n be a whole number, prove that $\frac{1}{6}(n^3 + 5n)$ is also a whole number.

7. If n be an even number, show that $n^3 + 20n$ is divisible by 48.

8. If the continued product of the first n odd numbers be divided by the continued product of the first n even numbers, prove that the quotient is a terminating decimal.

3.

TESTS OF PARALLELISM.

1. Prove that two straight lines are parallel : (i) if a transversal make a pair of alternate angles equal ; (ii) if they join the ends of equal and parallel straight lines towards the same parts ; (iii) if equivalent triangles with equal bases be placed

so that the base of either is on one line and the vertex on the other.

2. Prove, by using the first and second of the above tests, that the straight line which joins the mid-points of two sides of a triangle is parallel to the third side.

3. Prove the same theorem by using the third test.

4. $ABCDE$ is a regular pentagon; AC and BD meet in F ; show that $AEDF$ is a parallelogram.

5. ABC and DEF are two triangles such that AB is equal and parallel to DE , and AC to DF ; prove that BC is equal and parallel to EF .

6. Equivalent rectangles $ABDE$, $ACFG$ are described on the sides of a triangle right-angled at A and outside the triangle; show that EG is parallel to BC .

7. Prove that a straight line which divides two sides of a triangle, or those sides produced, in the same ratio is parallel to the third side.

8. ABC is a triangle right-angled at C , CN is drawn perpendicular to AB , the bisectors of the angles CAN , NCB meet CN , NB in E , F respectively; prove that EF is parallel to BC .

4.

TESTS OF PERPENDICULARITY.

1. ABC is an isosceles triangle in which $AB = AC$; if D be the mid-point of BC , show that AD is perpendicular to BC .

2. A transversal AC cuts the parallel straight lines AB , CD in A , C ; show that the bisectors of the angles BAC , ACD are perpendicular.

3. D is the mid-point of the base BC of a triangle ABC and is such that DA, DB, DC are all equal; show that AB is perpendicular to AC .

4. ABC is a triangle such that $BC^2 = AB^2 + AC^2$; show that AB is perpendicular to AC .

5. N is a point in the base BC of a triangle ABC such that $AB^2 - AC^2 = BN^2 - CN^2$; show that AN is perpendicular to BC .

6. $ABCD$ is a quadrilateral such that

$$AB^2 + CD^2 = BC^2 + AD^2;$$

show that AC is perpendicular to BD . Deduce from this the preceding example.

7. ABC is a triangle, BE, CF altitudes intersecting in O ; AD (produced if necessary) meets BC in D ; show that $\angle BAD = \angle FEB = \angle FCB$, and thence that AD is perpendicular to BC .

8. Draw up a list of the tests of perpendicularity contained in the above cases.

5.

THE ANGLES OF A POLYGON.

1. Show that the sum of the interior angles of any convex polygon of n sides is equal to $2n - 4$ right angles.

2. Two angles of a pentagon are 80° and 40° ; of the remaining angles, one is 10° more than the second and the second is 10° more than the third; find the number of degrees in each of the remaining angles.

3. Show that a tessellated pavement may be formed of tiles in the form of (i) an equilateral triangle, (ii) a square, and (iii) a regular hexagon, and of no other regular polygons of one form only, but (iv) of squares and regular octagons together.

4. If $ABCDEFGH$ be a regular heptagon, show that DF is parallel to AB .

5. If the alternate angles of a polygon of n sides be produced to meet, the angles so formed are together equal to $2n - 8$ right angles.

6. Any three angles of a convex pentagon are together greater than the angles of a triangle.

7. Any number of angles of a convex polygon are together greater than all the angles of a convex polygon of which the number of sides is the same as the number of angles of the first figure taken.

8. Three equiangular polygons, of m , n and p sides, have one vertex in common and just fill up the whole space at that vertex; show that

$$\frac{1}{m} + \frac{1}{n} + \frac{1}{p} = \frac{1}{2}.$$

6.

HIGHEST COMMON FACTOR AND LOWEST COMMON MULTIPLE.

1. If A and B represent two algebraical expressions arranged in descending powers of a common letter x , B being of not lower dimensions in x than A , and if, when B is divided by A , the quotient be Q and the remainder R , show that

$$R = B - AQ \quad \text{or} \quad B = AQ + R.$$

Hence, show that A and B have exactly the same common factors as R and A , and deduce the rule for finding the H.C.F. of A and B .

2. If H be the H.C.F. and L the L.C.M. of two algebraical expressions A and B , show that $HL = AB$.

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3. The G.C.M. of two numbers is 123 and their L.C.M. is 8856; find the numbers.

4. Find the greatest number which is such that, when 12288, 19139 and 28200 are divided by it, the remainders are all the same.

5. When 909365 is divided by a certain number, the three remainders are 1348, 1094 and 102 respectively; find the divisor and the quotient.

6. If $x^3 - ax^2 - bx + 1$ and $x^3 - bx^2 - ax + 1$ have a common algebraical factor, show that either $a + b = 2$ or that the expressions are identical.

7. If H be the H.C.F. of two expressions A and B , show, by applying the proof of the method of finding H , that two other expressions L and M can always be found such that $LA + MB = H$.

8. Apply the proof of the method of finding the H.C.F. in order to find two integral functions of x , P and Q , such that

$$(2x^3 + 5x^2 - 6x + 5)P - (x^2 + 3x - 2)Q = 1.$$

7.

ROOTS OF QUADRATIC EQUATIONS. I.

1. Solve the equation $ax^2 + bx + c = 0$, and, if a , b , c be real, deduce the conditions that the roots may be (i) real and different, (ii) real and equal, (iii) imaginary and different; and show that it is impossible for a quadratic equation with real coefficients to have a pair of roots equal and imaginary.

2. Find the value or values of a when the roots of the following equations are equal:

$$(i) \ 3x^2 - 7x + 4a = 0, \quad (ii) \ 4x^2 - (2 + a)x + 9 = 0;$$

and find the roots in each case.

3. Show that the roots of the equation (i) $ax^2 + 2x - a = 0$ are real, whatever real value a may have ; (ii) $x^2 - 6x + a = 0$ are real, if $a \geq 9$; (iii) $x^2 + (a - 3)x + 1 = 0$ are real, if a do not lie between 1 and 5.

4. If one of the equations

$$x^2 + x(2a - b) + ab = 0, \quad x^2 + x(4a - b) + 3a^2 = 0$$

have equal roots, so also has the other.

5. Prove that the roots of one of the equations

$$16a^2x^2 - 8a^2x + b^2 = 0, \quad 4a^2x^2 + 4b^2x + b^2 = 0$$

are real and of the other imaginary.

6. If the roots of the equation

$$q^2x^2 + (p^2 - 2q)x + q + 1 - \frac{p^2}{4} = 0$$

be equal, prove that $p^2 = 4q$.

7. If the roots of the equation

$$(x + b)(x + c) + (x + c)(x + a) + (x + a)(x + b) = 0$$

be equal, show that $a = b = c$.

8. If the roots of the equation $x^2 + px + q = 0$ be equal, prove that one root of the equation

$$ax^2 + p(a + b)x + q(a + 2b) = 0$$

is equal to each, and find the other root.

8.

RATIO. I.

1. If $ax + by + cz = 0$ and $a'x + b'y + c'z = 0$, prove that

$$\frac{x}{bc' - b'c} = \frac{y}{ca' - c'a} = \frac{z}{ab' - a'b}.$$

2. Hence, solve the equations $3x + 2y + 8 = 0$, $5x + 3y + 13 = 0$.

3. Solve the equations

$$2x - 3y + z = 0, \quad x - 6y + 3z = 0, \quad xyz = 5.$$

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4. Solve the equations

$$ax + cy + bz = cx + by + az = bx + ay + cz = a^3 + b^3 + c^3 - 3abc.$$

5. If $ay - bx = cx - az = bz - cy$, prove that

$$x/a = y/b = z/c,$$

provided that $a + b + c$ be not equal to zero.

6. If $ax + by + cz = 0$ and $a^2x + b^2y + c^2z = 0$, show that $a^3x + b^3y + c^3z$ cannot vanish unless two of the quantities a, b, c be equal.

7. Eliminate x from the equations

$$a + c = \frac{b}{x} - dx, \quad a - c = \frac{d}{x} - bx.$$

8. Eliminate x, y, z from the equations

$$\frac{y-z}{y+z} = a, \quad \frac{z-x}{z+x} = b, \quad \frac{x-y}{x+y} = c.$$

9.

PYTHAGORAS' THEOREM AND ITS EXTENSIONS.

1. $ABCD, AEFG$ are two squares with the sides BA, AE in a straight line, the squares being on opposite sides of this line; CB, FG are produced to meet in H , and CD, FE in K ; show that the figure $CKFH$ is a square. Also, if BG, DE be joined and the triangles HBG, GAB, EAD, DEK be placed round the square $CKFH$ so that their right angles coincide with those of the square, show that the figure formed by their hypotenuses is a square.

Hence, prove Pythagoras' theorem.

2. ABC is a triangle right-angled at A , and on its sides the squares $BCDE, CAFG, ABHK$ are described. The square $ABHK$ is divided into four congruent quadrilaterals by lines drawn through its centre parallel and perpendicular to BC ;

show that these four quadrilaterals together with the square $CAFG$ may be fitted together so as to make up the square $BCDE$.

3. ABC is any triangle, $BCDE$, $CAFG$, $ABHK$ are squares described on the sides outside the triangle; the altitudes AL , BM , CN are produced to meet the opposite sides of the squares in X , Y , Z ; show that

rect. AY = rect. AZ , rect. BZ = rect. BX , rect. CX = rect. CY .

4. Hence, show that, in an obtuse-angled triangle, the square described on the side opposite the obtuse angle is greater than the sum of the squares on the sides containing it by twice the rectangle contained by either of those sides and the projection of the other upon it; and that, in any triangle, the square on the side opposite an acute angle is less than the sum of the squares on the sides containing it by twice the rectangle contained by either of those sides and the projection of the other upon it.

5. (i) One angle of a triangle is 120° , and the sides containing it are 10 and 7 ins. long; find the length of the side opposite the obtuse angle. (ii) One angle of a triangle is 60° , and the sides containing it are 11 and 15 ins. long; find the length of the side opposite the angle of 60° . (iii) The three sides of a triangle are 15, 11 and 9 ins. long; find the length of the projection of the second side upon the third.

6. Prove, without assuming the theorem in No. 7, that the equilateral triangle described on the hypotenuse of a right-angled triangle is equal to the sum of the equilateral triangles described on the sides containing the right angle.

7. If similar and similarly situated polygons be described on the three sides of a right-angled triangle, show that the area of the polygon described on the hypotenuse is equal to

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the sum of the areas of those described on the sides containing the right angle.

8. Hence, give a geometrical construction for a circle equal in area to the sum of two given circles.

10.

APOLLONIUS' THEOREM.

1. The sum of the squares on two sides of a triangle is double the sum of the squares on half the third side and on the median to that side.

2. Find the lengths of the medians of a triangle, the sides of which are 10, 11, 12 ins.

3. Find the locus of a point which moves so that the sum of the squares on its distances from two fixed points is constant.

4. The sum of the squares on the sides of a parallelogram is equal to the sum of the squares on the diagonals.

5. If $ABCD$ be a rectangle and O any point in its plane, show that $OA^2 + OC^2 = OB^2 + OD^2$.

6. The sum of the squares on the sides of a quadrilateral is equal to the sum of the squares on the diagonals together with four times the square on the line joining the mid-points of the diagonals.

Hence, show that a quadrilateral is a parallelogram if the sum of the squares on the sides be equal to the sum of the squares on the diagonals.

7. D, E are the points of trisection of the side BC of the triangle ABC ; show that $AB^2 + AC^2 = AD^2 + AE^2 + 4 \cdot DE^2$.

8. If P be a point in the side BC of a triangle ABC so that $m \cdot BP = n \cdot PC$, prove that

$$mc^2 + nb^2 = m \cdot BP^2 + n \cdot CP^2 + (m+n)AP^2.$$

11.

MEDIAL SECTION.

1. Divide a straight line in medial section, that is, so that the rectangle contained by the whole line and one part may be equal to the square on the other part.

2. If a be the length of the given line, and x that of the latter part, find the quadratic equation giving x in terms of a , and show that $x = \frac{1}{2}(-a \pm a\sqrt{5})$.

Also, show that, approximately, the values of x/a are respectively $\cdot 6$ and $-1\cdot 6$.

3. Show that these values correspond to the points of internal and external medial section respectively, and give the geometrical construction for the latter point.

4. If a straight line be divided in medial section, show that the two parts of the line are in the ratio $3 - \sqrt{5} : \sqrt{5} - 1$.

5. If a straight line AB be divided internally at H in medial section, show that

$$(i) AB^2 + BH^2 = 3 \cdot AH^2; \quad (ii) (AB + BH)^2 = 5 \cdot AH^2.$$

6. If a straight line be divided in medial section, the rectangle contained by the sum and difference of the parts is equal to the rectangle contained by the parts.

7. If AB be divided at H in medial section, and if X be the mid-point of the greater segment AH , the triangle with sides equal to AH , XH , BX is right-angled.

8. Also, show that one angle A of the triangle so formed is 18° , by proving that $\cos 2A = \sin 3A$.

12.

ROOTS OF QUADRATIC EQUATIONS. II.

1. If the roots of the equation $x^2 + ax = 8x + 6a$ be real, show that $a \geq 8$ and $a \leq -2$.

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2. Show that, if the roots of the equation

$$(a^2 + b^2)x^2 + 2(ac + bd)x + c^2 + d^2 = 0$$

be real, they will be equal, and find the equal roots.

3. If the sum of the roots of a quadratic equation be 2, and the sum of the cubes of the roots be 5, find the equation.

4. The equation $x^2 - px + q = 0$ has both its roots increased by a certain quantity, and thus becomes $x^2 + mx - m^2 = 0$; prove that $5m^2 = p^2 - 4q$.

5. In a certain quadratic equation the coefficients of x^2 and x are 1 and 2 respectively, and the addition of 8 to each of the roots changes the sign, but not the magnitude of the third term. Find the original equation and the coefficient of x in the transformed equation.

6. Show that one of the roots of the equation $px^2 + qx + r = 0$ will be double one of the roots of the equation $rx^2 + qx + p = 0$, if either $r = 2p$ or $2p + r = \pm q\sqrt{2}$.

7. The equations $x^2 + px + q = 0$, $x^2 + rx + s = 0$ have a common root; form the quadratic equation satisfied by the other roots.

8. If α be any root of the equation $x^3 - 3x = 1$, show that $2 - \alpha^2$ is also a root, and find the third root.

13.

THEORY OF QUADRATIC EXPRESSIONS.

1. If $(x+b)(x+c) + (x+c)(x+a) + (x+a)(x+b)$ be a complete square in x , show that $a = b = c$.

2. If $ax^2 + 2bx + c$ be a perfect square, show that the product of $ax^2 + 2bx + c$ and $ay^2 + 2by + c$ is the square of $axy + b(x+y) + c$.

3. If a, b be the roots of $x^2 + px + q = 0$ and $a > c > b$, prove that $c^2 + pc + q$ is negative.

4. Find the conditions that $ax^2 + bx + c$ should be positive for all real values of x ; and show that $x^2 - (p + q)x + p^2 - pq + q^2$ is positive for all real values of x .

5. If $y = a + bx + cx^2$, show that y has a maximum or minimum value according as the sign of c is negative or positive, and that the corresponding value of x is one-half the sum of the roots of the equation $a + bx + cx^2 = 0$.

6. Find the maximum value of the expression $3 + 6x - x^2$ and the minimum value of the expression $x^2 + 8x + 20$; and illustrate both results by drawing the graphs of the expressions in the neighbourhood of their turning values.

7. The graph of the equation $y = a + bx + cx^2$ passes through the points $(0, 2)$, $(1, 6)$, and the maximum value of y occurs when $x = 2\frac{1}{2}$; find the values of a, b, c .

8. A horse can draw one-third of a ton of coal a certain distance in a given time; if his load exceed this, the time he takes is increased by a quantity proportional to the square of the excess. When his load is four-thirds of a ton he takes four times as long as at first. Show that he is most efficient when his load is two-thirds of a ton.

14.

REDUCTION OF EXPRESSIONS TO THE SUM OF TWO OR MORE SQUARES.

1. Express $a^2 + b^2 + c^2 - bc - ca - ab$ as half the sum of three squares, and hence show that, if a, b, c be different real quantities, (i) $a^2 + b^2 + c^2 > bc + ca + ab$; and (ii) the given expression is not altered if a, b, c be increased or decreased by the same quantity.

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2. Prove that $(a^2 + b^2 + c^2)(x^2 + y^2 + z^2) - (ax + by + cz)^2$ can be expressed as the sum of three squares.

3. Express $2(x - y)(x - z) + 2(y - z)(y - x) + 2(z - x)(z - y)$ as the sum of three squares.

4. Show that (i) $(a^2 + b^2)(c^2 + d^2)$ can be expressed as the sum of two squares in two different ways, and (ii) that

$$(a^2 + b^2)(c^2 + d^2)(e^2 + f^2)$$

can be expressed as the sum of two squares in four ways.

5. If x, y, z, a, b, c be all real, and if

$$x^2 + y^2 + z^2 + a^2 + b^2 + c^2 = 2(ax + by + cz),$$

prove that $x = a, y = b, z = c$.

6. If a, b, c, x, y, z be real quantities, and if

$$(a + b + c)^2 = 3(bc + ca + ab - x^2 - y^2 - z^2),$$

prove that $a = b = c$ and $x = y = z = 0$.

7. Show that

$$9x^2 + 4y^2 - 30ax + 20by + 25(a^2 + b^2)$$

can be expressed as the sum of two squares, and hence show that if the above expression be equal to zero, $x = 5a/3$ and $y = -5b/2$.

8. Find the least value which the expression

$$4x^2 + y^2 - 12x + 8y + 30$$

can assume for real values of x and y .

15.

PERIODS OF CIRCULAR FUNCTIONS.

1. Defining the *period* of a function $f(\theta)$ as the least value of α for which $f(\theta + \alpha) = f(\theta)$ for all values of θ , show that the cosine, sine, secant and cosecant of θ are periodic functions of θ with period 2π , and that the tangent and cotangent of θ are periodic functions of θ with period π .

2. Find the periods of $\sin 3\theta$, $\tan \frac{\theta}{2}$ and $\cos (\lambda\theta + \alpha)$.
3. Show that $a \sin (\theta + \alpha) + b \sin (\theta + \beta)$ can be expressed as a single circular function of the same period.
4. Find the period of (i) $\cos \theta - \sin \theta$, (ii) $\cot \theta - \tan \theta$.
5. Prove that the product of two circular functions of the same period is a function of half the period.
6. If the displacement s of a particle from its position of rest at time t be represented by the equation $s = a \sin (\lambda t + \epsilon)$, show that the velocity of the particle is $a\lambda \cos (\lambda t + \epsilon)$ and the acceleration $-a\lambda^2 \sin (\lambda t + \epsilon)$.
7. Hence, if T be the period of any one of these expressions, show that the maximum velocity is $2\pi a/T$ and the maximum acceleration $4\pi^2 a/T^2$.
8. During the Tokyo earthquake of June 20, 1894, the maximum amplitude of the motion was 36.5 mm., and the period of the corresponding vibration was 1.8 secs.; find the maximum velocity and the maximum acceleration.

16.

THE EQUATION $a \cos \theta + b \sin \theta = c$.

1. Explain how to solve the equation

$$a \cos \theta + b \sin \theta = c,$$
 showing that, for real roots, $c^2 \geq a^2 + b^2$. If this condition be satisfied, prove that the equation is satisfied by two positive values of θ less than 2π .
2. Show that the roots of the equation

$$a \cos \theta + b \sin \theta = c$$
 may be obtained by the following geometrical construction :
 From any line Ox , cut off a part OA equal to a in magnitude

and sense; draw AB perpendicular to OA and equal to b in magnitude and sense; with O as centre and OB as radius, describe a circle; from OB , cut off ON equal to c in magnitude and sense, and through N draw the chord PP' perpendicular to OB ; then all the angles bounded by Ox and OP , and by Ox and OP' , satisfy the given equation.

3. Solve the equation

$$\cos \theta + \sqrt{3} \sin \theta = \sqrt{2},$$

and show that the general solution may be expressed in either of the forms

$$2n\pi + \frac{\pi}{3} \pm \frac{\pi}{4} \quad \text{or} \quad n\pi - \frac{\pi}{6} + (-1)^n \frac{\pi}{4},$$

where n is any integer. Also, show that these formulae give precisely the same series of angles.

4. Solve the equation

$$4 \cos \theta + 5 \sin \theta = 6.$$

5. If α, β be the two roots of the equation

$$a \cos \theta + b \sin \theta = c,$$

which lie between 0 and 2π , find the values of $\tan \frac{\alpha}{2} + \tan \frac{\beta}{2}$ and $\tan \frac{\alpha}{2} \tan \frac{\beta}{2}$.

6. If α, β be two values of θ , not equal nor differing by a multiple of 2π , and satisfying the equation

$$a \cos \theta + b \sin \theta = c,$$

prove that

$$\cos (\alpha + \beta) = \frac{a^2 - b^2}{a^2 + b^2}.$$

7. Prove also that

$$\sec^2 \frac{\alpha - \beta}{2} = \frac{a^2 + b^2}{c^2}.$$

8. If $\alpha, \beta, \alpha + \beta$ be three different solutions of the equation

$$a \cos \theta + b \sin \theta = c,$$

prove that $a = c$.

17.

EXPRESSION OF $\cos \alpha$ AND $\sin \alpha$ IN TERMS OF $\sin 2\alpha$.

1. Find $\cos \alpha$ and $\sin \alpha$ in terms of $\sin 2\alpha$.

2. If $\cos \alpha$ and $\sin \alpha$ be found in terms of $\sin 2\alpha$, show algebraically that there should be four different values of each.

3. Prove the same theorem geometrically.

4. In the equations

$$\cos \alpha + \sin \alpha = \pm \sqrt{1 + \sin 2\alpha},$$

$$\cos \alpha - \sin \alpha = \pm \sqrt{1 - \sin 2\alpha},$$

show that the signs to be taken before the radicals are $++$ if α lie between $2n\pi - \frac{\pi}{4}$ and $2n\pi + \frac{\pi}{4}$, $+-$ if between $2n\pi + \frac{\pi}{4}$ and $2n\pi + \frac{3\pi}{4}$, $--$ if between $2n\pi + \frac{3\pi}{4}$ and $2n\pi + \frac{5\pi}{4}$, and $-+$ if between $2n\pi + \frac{5\pi}{4}$ and $2n\pi + \frac{7\pi}{4}$.

5. Find the values of $\cos 9^\circ$, $\sin 52\frac{1}{2}^\circ$, $\sin 97\frac{1}{2}^\circ$ and $\cos 195^\circ$.

6. Find the limits between which 2α must lie when

$$2 \sin \alpha = -\sqrt{1 + \sin 2\alpha} + \sqrt{1 - \sin 2\alpha}.$$

7. Find the general values of the limits between which α lies when $\sin^2 \alpha > \cos^2 \alpha$.

8. Show that

$$2 \cos \frac{\alpha}{2} = (-1)^m \sqrt{1 + \sin \alpha} + (-1)^n \sqrt{1 - \sin \alpha},$$

where m and n are the greatest integers contained in

$$\frac{\alpha + 90}{360} \text{ and } \frac{\alpha + 270}{360} \text{ respectively,}$$

and α is measured in degrees.

18.

**AMBIGUOUS CASE IN THE SOLUTION OF
TRIANGLES.**

1. If the elements a , b , A of a triangle be given, find the value or values of c by means of the equation

$$a^2 = b^2 + c^2 - 2bc \cos A,$$

and consider the different cases that may arise according as a is greater than, equal to or less than b .

Also, if $a = b \tan A$, show that the two possible values of c are $b \sec A$ and $b \cos 2A \sec A$.

2. In a triangle, $a = 27$, $b = 38$, $A = 30^\circ$; find the two values of the side c .

3. If, in a triangle, a , b and A be given, and if c_1 , c_2 be the values of the third side, show that $c_1 c_2 = b^2 - a^2$.

4. Show also that

$$c_1^2 - 2c_1 c_2 \cos 2A + c_2^2 = 4a^2 \cos^2 A.$$

5. If, in a triangle in which a , b , A are given, the values of C differ by a right angle, show that $a^2 = 2b^2 \sin^2 A$.

6. In the ambiguous case, show that the two triangles have circumcircles of the same radius, and find the distance between the circumcentres.

7. In the ambiguous case, the sum of the radii of the two incircles and of the two excircles opposite the given angle is equal to twice the common altitude of the two triangles.

8. In the ambiguous case, if a , b , A be given, and if S , S' be the areas of the two triangles, the continued product of the radii of the incircles and of the excircles opposite the angle B is equal to SS' .

19.

THE PEDAL TRIANGLE OF A TRIANGLE.

1. Each angle of the pedal triangle of a given triangle is bisected by the corresponding altitude.

2. In an acute-angled triangle, each angle of the pedal triangle is equal to the supplement of twice the opposite angle of the given triangle.

What does this theorem become when the triangle is obtuse-angled ?

3. The angular points of a triangle are the excentres of the pedal triangle.

4. If S be the circumcentre of a triangle, show that SA , SB , SC are perpendicular to the sides of the pedal triangle.

5. The radius of the circumcircle of a given triangle is twice that of the circumcircle of the pedal triangle.

6. The sides of the pedal triangle of a given triangle are $R \sin 2A$, $R \sin 2B$, $R \sin 2C$, and the perimeter

$$4R \sin A \sin B \sin C.$$

7. The area of the pedal triangle is $2S \cos A \cos B \cos C$, where S is the area of the given triangle.

8. The radius of the incircle of the pedal triangle is $2R \cos A \cos B \cos C$; find corresponding expressions for the radii of the excircles.

20.

THE AREA OF A TRIANGLE.

1. Find the area of a triangle in terms of any two sides and the included angle.

2. Find the area of a triangle in terms of the sides.

3. Find the area of a triangle in terms of two angles and the adjacent side. If $B=30^\circ$, $C=45^\circ$ and $a=6$ ins., find the area of the triangle.

4. Show that the area of a triangle is

$$(i) \ 2R^2 \sin A \sin B \sin C,$$

$$(ii) \ r^2 \cot \frac{A}{2} \cot \frac{B}{2} \cot \frac{C}{2}.$$

5. Show that the area of a triangle is

$$\frac{1}{4}(b^2 \sin 2C + c^2 \sin 2B),$$

and interpret the result geometrically.

6. Show that the area of a triangle is

$$\frac{1}{4}(a^2 \cot A + b^2 \cot B + c^2 \cot C),$$

and interpret the result geometrically

7. Find the area of a triangle in terms of the coordinates of its angular points.

8. If l , m , n be the lengths of the medians of a triangle, show that the area of the triangle is

$$\frac{1}{3} \{ (l^2 + m^2 + n^2)^2 - 2(l^4 + m^4 + n^4) \}^{\frac{1}{2}}.$$

21.

RATIO. II.

1. If the equations $ax^2+bx+c=0$, $a'x^2+b'x+c'=0$ have a common root, show that

$$(ca' - c'a)^2 = (bc' - b'c)(ab' - a'b).$$

2. If the equations $x^2+px+q=0$, $x^2+p'x+q'=0$ have a common root, show that it is either

$$\frac{pq' - p'q}{q - q'} \quad \text{or} \quad \frac{q - q'}{p' - p}.$$

3. If the equations $ax^2 + 2bx + c = 0$, $cx^2 + 2bx + a = 0$ ($c \neq a$) have a common root, then the other roots are given by

$$ac(a+c)(x^2+1) + 2b(c^2+a^2)x = 0.$$

4. If
$$\frac{2x-3y}{3z+y} = \frac{z-y}{z-x} = \frac{x+3z}{2y-3x},$$

prove that each fraction is equal to x/y , and hence show that either $x=y$ or $z=x+y$.

5. If, in a triangle,

$$\frac{b+c}{4} = \frac{c+a}{5} = \frac{a+b}{6},$$

show that

$$\frac{\sin A}{7} = \frac{\sin B}{5} = \frac{\sin C}{3}$$

and

$$\frac{\cos A}{-7} = \frac{\cos B}{11} = \frac{\cos C}{13}.$$

6. If $bz^2 + cy^2 = ayz$, $cx^2 + az^2 = bzx$, $ay^2 + bx^2 = cxy$, prove that

$$x^3 + y^3 + z^3 + xyz = 0.$$

7. If

$$\frac{by+cz}{b+c-a} = \frac{cz+ax}{c+a-b} = \frac{ax+by}{a+b-c}$$

and

$$ax+by+cz = a+b+c,$$

prove that $x = 3 - \frac{b+c}{a}$, with corresponding values for y and z .

8. Show that the eliminant of the equations

$$x+y+z=0, \quad \frac{x^2}{a} + \frac{y^2}{b} + \frac{z^2}{c} = 0, \quad ayz + bzx + cxy = 0$$

is $(b+c)(c+a)(a+b) = a^3 + b^3 + c^3 + 5abc$.

22.

SIMULTANEOUS EQUATIONS WITH INFINITE SOLUTIONS.

1. What is the nature of the solution of the equations

$$2x+3y=12, \quad 4x+6y=24 ?$$

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Also of the equations

$$2x + 3y = 12, \quad 4x + 6y = 30 ?$$

Illustrate both cases by graphs.

2. What is the nature of the solution of the equations

$$5x + 3y + 2z = 3, \quad 4x + 2y + 3z = 2, \quad 3x + y + 4z = 4 ?$$

3. Show that the equations

$$ax + by + c = 0, \quad a'x + b'y + c' = 0$$

have (i) infinite solutions if $ab' - a'b = 0$, (ii) the value of x zero if $bc' - b'c = 0$ and the value of y zero if $ca' - c'a = 0$.

What will be the nature of the solution if $ab' - a'b = 0$ and $bc' - b'c = 0$?

4. Solve the equations

$$x^2 - y^2 = 7, \quad x - y = 1,$$

and illustrate the solution by means of graphs.

5. Solve the equations

$$y^2 - xy = 2, \quad y^2 - x^2 = 3,$$

and illustrate the solution by means of graphs.

6. Show that the equations

$$ax^2 + 2hxy + by^2 = c, \quad a'x^2 + 2h'xy + b'y^2 = c'$$

have two infinite solutions if each of the expressions

$$ax^2 + 2hxy + by^2 \quad \text{and} \quad a'x^2 + 2h'xy + b'y^2$$

have a factor of the form $y - \mu x$.

7. What is the nature of the solutions of the equations

$$x^2 - xy = 4, \quad x^2 - xy = 8 ?$$

Illustrate the solution by means of graphs.

8. Show that the equations

$$x^3 + y^3 = 3axy \quad \text{and} \quad x + y = c$$

have one infinite and two complex solutions if $c > -a$, and two real and one infinite solutions if c lie between $-a$ and $3a$. What is the nature of the solutions if $c > 3a$?

23.

GRAPHS AND THE SOLUTION OF ALGEBRAICAL EQUATIONS.

1. Solve the equations

$$x - y = 2, \quad x^2 - y^2 = 8,$$

showing that the solutions are

$$x = 3, \quad y = 1;$$

and a pair of infinite roots.

Illustrate both solutions by means of the graphs of the equations.

2. Solve the equations

$$x + y = 0, \quad x^2y^2 + 2xy - 15 = 0,$$

and illustrate the method of solution by means of graphs.

3. Draw the graph of $y = \frac{x+5}{x-3}$;

find the equations of its asymptotes, and determine on which side of the asymptotes the corresponding branches of the curve lie.

4. Draw the graph of

$$y = \frac{(x-2)(x-5)}{(x-1)(x-3)}.$$

5. Find for what value of k the graph of $2x + 3$ will touch the graph of

$$\frac{2x+3}{x+k}.$$

Trace the last-mentioned graph if k have the required value.

6. Trace the graphs of $x^2/4$ and $3/x$ between the limits 1 and 4 (the unit of length being not less than one inch).

Hence, find an approximation to the cube root of 12.

7. Draw the graph of $y = x + \frac{2}{x}$,

taking one inch as the unit. Hence, solve the equation

$$x + \frac{2}{x} = -4.$$

Verify the result by an accurate algebraical solution of the equation.

8. Draw the graph of the equation

$$y^2 = (x-a)(x-b)(x-c),$$

where a, b, c are positive and in ascending order of magnitude ; and trace the changes in the forms which the curves assume near the axis of x when

$$(i) \ a=b, \quad (ii) \ b=c, \quad (iii) \ a=b=c.$$

24.

LIMITS OF ALGEBRAICAL EXPRESSIONS.

1. If x be real, show that

$$\frac{x^2 - 5x - 5}{x - 6}$$

cannot lie between 5 and 9 ; and illustrate the result by drawing the graph of the above expression.

2. If x be real, show that

$$\frac{x^2 - 2x + 1}{x^2 + 2x + 2}$$

is not greater than 5 and not less than 0, and illustrate the result by drawing the graph of the expression.

3. If x be real, show that

$$\frac{x^2 - 6x + 5}{x^2 - 9x + 18}$$

may assume any real value ; and illustrate the result by drawing the graph of the expression.

4. Show that real values of x and y can be found which will satisfy the equation

$$4x^2 + 2xy + y^2 - 39x - 6y + 99 = 0,$$

and that the values of x lie between 5 and 6, and those of y between $-1\frac{1}{2}$ and $-3\frac{1}{2}$.

5. Find the limits within which x must lie in order that

$$\frac{5x^2 - 11x + 3}{2x^2 - 9x + 4}$$

may be less than unity.

6. Find the values of a, b so that the limits of the expression

$$\frac{ax^2 + 2bx + 1}{x^2 + x + 1},$$

corresponding to real values of x , may be 1 and $\frac{1}{5}$.

7. Show that the expression

$$\frac{A}{x - \alpha} + \frac{B}{x - \beta}$$

admits of all values for real values of x if A and B have the same sign; but that there is a range of values of extent

$$\frac{4\sqrt{(-AB)}}{\alpha - \beta},$$

which the expression cannot have, if A and B have opposite signs and if α be unequal to β .

8. If three real quantities satisfy the equations

$$x + y + z = 5 \quad \text{and} \quad yz + zx + xy = 8,$$

prove that none of them is less than 1 nor greater than $2\frac{1}{3}$.

25.

THE ALTITUDES OF A TRIANGLE.

1. The altitudes of a triangle are concurrent.

2. O is the orthocentre of the triangle ABC , and the altitude AD is produced to meet the circumcircle in X ; prove that $OD = DX$.

3. If O be the orthocentre of the triangle ABC , show that OA is double of the perpendicular from the circumcentre to BC .

4. Each altitude of a triangle is a harmonic mean between the radii of two excircles.

5. The sum of the reciprocals of the altitudes of a triangle is equal to the sum of the reciprocals of the radii of the excircles.

6. If the sides of a triangle be roots of the equation

$$x^3 - lx^2 + mx - n = 0,$$

show that the altitudes are the roots of the equation

$$8R^3x^3 - 4mR^2x^2 + 2lnRx - n^2 = 0.$$

7. ABC is a triangle right-angled at A , and squares $BCDE$, $CAFG$, $ABHK$ are described on the sides outside the triangle; also AL is drawn perpendicular to DE ; show that AL , BG , CH are concurrent.

8. Through a point F in the diagonal BD of a square $ABCD$, lines are drawn parallel to the sides to meet AB in G , BC in E , CD in K , and DA in H . Prove that BH , CF , DG are concurrent.

26.

THE MEDIANS OF A TRIANGLE.

1. Show that the medians of a triangle are concurrent, (i) by geometry; (ii) by finding the centre of gravity of equal masses placed at the angular points.

2. Find the lengths of the medians in terms of the sides, and show that four times the sum of the squares on the medians is equal to three times the sum of the squares on the sides.

Find the lengths of the medians, the lengths of the sides being 8, 10 and 12 inches.

3. Prove that any two sides of a triangle are together greater than twice the median to the third.

4. Any two medians of a triangle are together greater than the third.

5. If AX , BY , CZ be the medians of a triangle, show that a triangle can be constructed with its sides equal and parallel to AX , BY , CZ .

6. Construct a triangle, being given the lengths of the three medians.

7. If θ , ϕ , ψ be the angles which the medians make with the sides to which they are drawn, show that

$$2 \cdot \cot \theta = \cot B + \cot C,$$

and hence that $\cot \theta + \cot \phi + \cot \psi = 0$.

8. Show that the angle between the medians drawn from B and C is $\cot^{-1} \frac{b^2 + c^2 - 5a^2}{12S}$. Hence, show that, if θ , ϕ , ψ

be the angles between the medians x , y and z ,

$$\cot \theta + \cot \phi + \cot \psi = -\frac{a^2 + b^2 + c^2}{4S} = -\frac{x^2 + y^2 + z^2}{3S},$$

and that $\Sigma \cot \theta + \Sigma \cot A = 0$.

27.

THE BISECTORS OF THE ANGLES OF A TRIANGLE.

1. Show that the bisectors of the angles of a triangle are concurrent.

2. Show that the length of the bisector of the angle A is

$$\frac{2bc \cos \frac{A}{2}}{b+c}.$$

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Find the corresponding expression for the length of the bisector of the external angle A .

3. Prove that the bisector of the angle A is equal to

$$\sqrt{\{bc(a+b+c)(b+c-a)\}/(b+c)}.$$

4. The sides of a triangle are 15, 13 and 11 ; find the length of the bisector of the largest angle.

5. Perpendiculars are drawn to the bisector of the vertical angle of a triangle from the ends of the base ; prove that the feet of the perpendiculars are equidistant from the mid-point of the base.

6. The three sides of a triangle are together greater than the three bisectors of the angles of the triangle.

7. Show that the greatest of the bisectors of the angles of a triangle is that which bisects the least angle.

8. If the bisectors of the angles A, B, C of a triangle ABC meet the opposite sides in D, E, F , and if the angles ADB, BEC, CFA be α, β, γ respectively, prove that

$$a \sin 2\alpha + b \sin 2\beta + c \sin 2\gamma = 0.$$

28.

THE NINE-POINTS CIRCLE.

1. ABC is a triangle, O its orthocentre, A', B', C' the mid-points of the sides, D, E, F the feet of the altitudes, L, M, N the mid-points of OA, OB, OC ; prove that each of the angles $MA'N, MDN$ is the supplement of the angle MLN ; and hence show that a circle passes through the nine points $L, M, N, D, E, F, A', B', C'$.

2. If S be the circumcentre and if SO cut $A'L$ in V , show that V is the mid-point of SO and of $A'L$, that $VL = \frac{1}{2} \cdot SA$,

and hence that the circle LDA' passes also through M, B', E, N, C', F .

3. Show that $B'C'MN$, $C'A'NL$, and $A'B'LM$ are rectangles, and therefore that $A'L$, $B'M$, $C'N$ are equal to one another and bisect one another.

4. Show that: (i) $\angle FBO = \angle FDO$, (ii) $\angle FMO = \angle FDE$, (iii) $\angle FA'E = \angle FMO$.

5. Bisect SO at V , draw any radius SP of the circumcircle, draw VQ parallel to SP , meeting OP in Q , and show that the locus of Q is a circle with centre V and radius half of SP . This circle obviously passes through L, M, N, D, E, F . Draw the radius SR parallel to VA' , and show that $OA'R$ is a straight line with its mid-point at A' .

6. Show that $\angle AEF = \angle FC'B'$, and thus that the points B', C', F, E are cyclic, and similarly that the points A', D lie on the same circle; again, show that $\angle LFA = \angle LB'C'$, and therefore that the circle passes through L , and similarly through M and N .

7. Show that each of the angles $A'ML$, $A'B'L$, $A'FL$ is a right angle, and hence that the circle on $A'L$ as diameter passes through M, B', F , and also through D, N, B', E .

8. Summarise the methods of proof outlined above.

29.

THE CIRCULAR FUNCTIONS OF 18° AND ALLIED ANGLES.

1. From the equation

$$\cos 2 \cdot 18^\circ = \sin 3 \cdot 18^\circ,$$

find $\sin 18^\circ$.

2. ABC is an isosceles triangle in which each of the angles B, C is double of the angle A . From AB , AD is cut off equal to BC . Show that $AD=CD$, and that BC touches the circle ACD . Hence, show that AB is divided at D in medial section.

Also, if the bisector AE of the angle A meet the base BC in E , show that the angle BAE is 18° , and hence show that

$$\sin 18^\circ = (\sqrt{5} - 1)/4.$$

3. From the formula $\sin 3\alpha = 3 \sin \alpha - 4 \sin^3 \alpha$, show that

$$\sin 54^\circ = (\sqrt{5} + 1)/4;$$

hence, show that $\sin 18^\circ$ and $-\sin 54^\circ$ are the roots of the equation

$$4x^2 + 2x = 1.$$

4. Show that

$$(i) \sin 54^\circ = \sin 18^\circ + \sin 30^\circ,$$

$$(ii) \cos(54^\circ - \alpha) - \cos(54^\circ + \alpha) - \cos(18^\circ - \alpha) \\ + \cos(18^\circ + \alpha) = \sin \alpha.$$

5. Two parallel chords of a circle, lying on the same side of the centre, subtend angles of 72° and 144° at the centre; show that the distance between them is equal to half the radius of the circle.

6. $ABCDE$ is a regular pentagon, and O any point on the minor arc AE of the circumcircle; show that

$$OA + OC + OE = OB + OD.$$

7. Two sides of a triangle are in the ratio $3 - \sqrt{5} : 2$, and the angles opposite them are in the ratio $1 : 3$; find the angles.

8. If $\cos \alpha = \tan \beta$ and $\cos \beta = \tan \alpha$, prove that

$$\cos^2 \alpha = 2 \sin 18^\circ.$$

30.

THE IDENTITY $\sin(\alpha + \beta) \sin(\alpha - \beta) = \sin^2 \alpha - \sin^2 \beta$.

1. Prove that $\sin(\alpha + \beta) \sin(\alpha - \beta) = \sin^2 \alpha - \sin^2 \beta$, and find the corresponding formula for $\cos(\alpha + \beta) \cos(\alpha - \beta)$.

2. (i) In any triangle, show that

$$\sin(A - B) : \sin C = a^2 - b^2 : c^2.$$

(ii) Show that

$$4 \sin \alpha \sin\left(\frac{\pi}{3} + \alpha\right) \sin\left(\frac{\pi}{3} - \alpha\right) = \sin 3\alpha.$$

3. Show that $\sin^2 42^\circ - \sin^2 12^\circ = (\sqrt{5} + 1)/8$.

4. Solve the equation $\sin^2(m + 1)\theta - \sin^2 \theta = \sin^2(m - 1)\theta$.

5. Prove geometrically that

$$\sin(\alpha + \beta) \sin(\alpha - \beta) = \sin^2 \alpha - \sin^2 \beta.$$

6. Prove that $\sin \theta = 2 \sin \frac{\theta}{2} \sin \frac{\theta + \pi}{2}$,

and hence that

$$\sin \theta = 2^{n-1} \sin \frac{\theta}{n} \sin \frac{\theta + \pi}{n} \sin \frac{\theta + 2\pi}{n} \dots \sin \frac{\theta + (n-1)\pi}{n},$$

where n is a positive integral power of 2.

7. In the preceding expression, show that the middle of the series of factors

$$\sin \frac{\theta + \pi}{n} \sin \frac{\theta + 2\pi}{n} \dots \sin \frac{\theta + (n-1)\pi}{n} \text{ is } \cos \frac{\theta}{n}.$$

8. Hence, show that

$$\begin{aligned} \sin \theta = 2^{n-1} \sin \frac{\theta}{n} \cos \frac{\theta}{n} & \left(\sin^2 \frac{\pi}{n} - \sin^2 \frac{\theta}{n} \right) \left(\sin^2 \frac{2\pi}{n} - \sin^2 \frac{\theta}{n} \right) \\ & \dots \left(\sin^2 \frac{n-2\pi}{n} - \sin^2 \frac{\theta}{n} \right). \end{aligned}$$

Deduce, from this equation, that

$$n = 2^{n-1} \sin^2 \frac{\pi}{n} \sin^2 \frac{2\pi}{n} \dots \sin^2 \frac{(n-2)\pi}{n}.$$

31.

THE CIRCLE OF APOLLONIUS.

1. If a point moves so that its distances from two fixed points are in a constant ratio, show that it lies on a circle.

2. If any point be taken on this circle, show that its distances from the two fixed points are in the constant ratio.

3. If the constant ratio be λ and the distance between the fixed points be a , find the radius of the circle of Apollonius.

4. ABC is a triangle ; find a point which is such that its distances from the points A, B, C are in the ratio $3 : 4 : 5$.

5. If A, B be the fixed points, and PBQ any chord of the circle of Apollonius, prove that AB bisects the angle PAQ .

6. By means of the circle of Apollonius, describe a triangle on a given base, with a given vertical angle, and with the sides containing that angle in a given ratio.

7. Show that the area of the greatest triangle of which the base is b , and the ratio of the other sides λ , is

$$\frac{1}{2} \frac{\lambda}{\lambda^2 - 1} b^2.$$

8. ABC is a triangle ; a point moves so that its distances from the points B, C are in the ratio $AB : AC$; a second point so that its distances from the points C, A are in the ratio $BC : BA$; a third point so that its distances from the points A, B are in the ratio $CA : CB$; prove that the three circles so described are concurrent.

32.

THE AREA OF A QUADRILATERAL.

1. Lines are drawn through the angular points of a quadrilateral parallel to the diagonals ; show that the area of the quadrilateral is half that of the parallelogram so formed.

2. If x, y be the lengths of the diagonals of a quadrilateral, and ϕ the angle between them, show that the area of the quadrilateral is $\frac{1}{2}xy \sin \phi$.

3. If a, b, c, d be the sides of a quadrilateral, and ϕ the angle between the diagonals, show that the area of the quadrilateral is

$$\frac{1}{4}(a^2 - b^2 + c^2 - d^2) \tan \phi.$$

4. If a circle can be inscribed in the quadrilateral, show that

$$Q = \frac{1}{2}(bd - ac) \tan \phi,$$

and hence, making use of the result in No. 2, that

$$Q = \frac{1}{2}\sqrt{\{x^2y^2 - (bd - ac)^2\}}.$$

5. Prove that the area of a cyclic quadrilateral is

$$\sqrt{\{(s-a)(s-b)(s-c)(s-d)\}}.$$

6. If, in any quadrilateral $ABCD$,

$$A = \omega + \alpha, \quad C = \omega - \alpha,$$

show that

$$(ad + bc) \cos \alpha \sin \omega - (bc - ad) \sin \alpha \cos \omega = 2Q.$$

7. By equating two expressions for BD^2 , show that

$$(bc - ad) \cos \alpha \cos \omega + (ad + bc) \sin \alpha \sin \omega = \frac{1}{2}(b^2 + c^2 - a^2 - d^2).$$

8. Squaring and adding the corresponding sides of the last two equations, show that

$$(ad + bc)^2 \sin^2 \omega + (bc - ad)^2 \cos^2 \omega = 4Q^2 + \frac{1}{4}(b^2 + c^2 - a^2 - d^2)^2,$$

and hence that

$$Q = \sqrt{\{(s-a)(s-b)(s-c)(s-d) - abcd \cos^2 \omega\}}.$$

Hence, show that, if the sides of a quadrilateral be given in length, its area is greatest when it is cyclic.

33.

ON CERTAIN LINES CONNECTED WITH A TRIANGLE.

1. Show that the line joining the feet of the altitudes from B, C meets BC produced at an angle $(C \sim B)$.

2. P is a point on the circumcircle of a triangle ABC ; show that the angle which the Simson line of P makes with BC is equal to the complement of the angle at the circumference which stands on the arc AP .

3. If O be the orthocentre of a triangle ABC , L and A' the mid-points of OA, BC , show that the angle $A'LD$ is equal to $(B \sim C)$.

4. The line joining the mid-points of BC and the altitude from A makes with BC an angle $\cot^{-1}(\cot B \sim \cot C)$.

5. The line joining the circumcentre and incentre of a triangle ABC makes with BC an angle

$$\tan^{-1} \frac{\cos B + \cos C - 1}{\sin B \sim \sin C}.$$

6. The line joining the circumcentre and orthocentre of a triangle ABC makes with BC an angle

$$\tan^{-1} \frac{\tan B \tan C - 3}{\tan B \sim \tan C}.$$

7. Prove that the circumcentre, centroid, nine-points centre and orthocentre of a triangle lie in order on a straight line (Euler's line).

8. Show that the Euler line forms an equilateral triangle with two of the sides if the angle between them be two-thirds of a right angle.

34.

THE PRINCIPLE OF PROPORTIONAL PARTS.

1. Explain the principle of proportional parts, and show that it is equivalent to regarding the curve of the function considered as a straight line between the two limiting values of the function given.

2. OA and OB are the bounding radii of a quadrant of a circle of radius 10 inches, and MP , an ordinate of the curve corresponding to a distance OM measured along the radius OA , is denoted by $\text{ord. } (OM)$; find $\text{ord. } (5.3)$, $\text{ord. } (5.34)$ and $\text{ord. } (5.4)$, each to five places of decimals.

3. From the values of $\text{ord. } (5.3)$ and $\text{ord. } (5.4)$, deduce that of $\text{ord. } (5.34)$ by the principle of proportional parts.

4. If n be any number and δ be small compared with n , show that

$$\log_{10}(n+\delta) - \log_{10}n = \mu \left\{ \frac{\delta}{n} - \frac{1}{2} \frac{\delta^2}{n^2} + \dots \right\},$$

where μ is the modulus of the common system of logarithms.

5. If $n \gg 10000$, $\delta = 1$, show that

$$\frac{\mu\delta^2}{2n^2} < .0000000025,$$

μ being .43429448; and, hence, if $\delta < 1$, show that

$$\log(n+1) - \log n : \log(n+\delta) : \log n = 1 : \delta$$

to at least seven places of decimals.

6. Show that

$$\sin(\theta + \delta) - \sin \theta = \delta \cos \theta - \frac{1}{2}\delta^2 \sin \theta - \frac{1}{6}\delta^3 \cos \theta + \dots$$

7. By considering the ratio of the second term to the first on the right-hand side of the preceding equation, show that, if δ be very small, the principle of proportional parts holds provided θ be not nearly $\frac{\pi}{2}$.

8. If θ be nearly $\frac{\pi}{2}$, show that the change in $\sin \theta$ does not vary directly as the change in the angle, that is, that the change in $\sin \theta$ is irregular.

Again, show that the second term $\frac{1}{2}\delta^2 \sin \theta$ is less than the first term $\delta \cos \theta$, and hence that, when θ is nearly $\frac{\pi}{2}$, the change in the sine is very small compared with δ , that is, the change in $\sin \theta$ is insensible.

35.

THE RADICAL AXIS TREATED GEOMETRICALLY.

1. If, from a point without two circles, two equal tangents be drawn, one to each circle, the given point lies on a straight line perpendicular to the line of centres. (This line is called the *radical axis* of the circles.)

2. If a , b be the radii of the circles and c the distance between their centres, show that the distances of the radical axis from the centres are

$$\frac{c^2 + a^2 - b^2}{2c} \quad \text{and} \quad \frac{c^2 - a^2 + b^2}{2c}.$$

Hence, show that the radical axis lies nearer the circle of larger radius than to the other.

3. Prove the following geometrical constructions for the radical axis of two non-intersecting circles: (i) Draw any circle cutting one of the given circles in C , D , and the other in E , F ; let CD , EF produced meet in P ; then the line through P perpendicular to the line of centres is the radical axis; (ii) From A , B , the centres of the circles, draw tangents AY , BX to the circles; with A , B as centres and AY , BX as radii, describe circles intersecting in P , Q ; then PQ is the radical axis.

4. The radical axes of three circles (the centres of which are not collinear) taken in pairs are concurrent. (The point of concurrence is called the *radical centre* of the circles.)

5. If, from any point on the radical axis of two circles as centre, a circle be described with radius equal to the tangents drawn from it to the circles, this circle cuts the line of centres in two fixed points. (These points are called the *limiting points* of the circles.)

6. All circles drawn through the limiting points of two circles cut the circles orthogonally.

7. The six limiting points of three circles taken in pairs are cyclic. Hence, draw a circle so as to cut three circles (the centres of which are non-collinear) orthogonally.

8. The difference between the squares on the tangents drawn from any point to two circles is equal to twice the rectangle contained by the line joining their centres and the perpendicular from the point on the radical axis.

36.

THE RECTANGLE FORMED BY THE BISECTORS OF THE ANGLES OF A PARALLELOGRAM.

1. Prove that the quadrilateral formed by the bisectors of the interior angles of a parallelogram is a rectangle.

2. Prove that two opposite angular points of this rectangle lie outside, on two opposite sides of, or inside, the parallelogram, according as each of these opposite sides is greater than, equal to, or less than, twice the other pair of sides.

3. If a , b be the sides of the given parallelogram, and α the angle between them, show that the sides of the rectangle are

$$(a-b) \cos \frac{\alpha}{2} \text{ and } (a+b) \sin \frac{\alpha}{2}.$$

4. Hence, show that, if the lengths of the sides of the parallelogram be given, the area of the rectangle is greatest when the parallelogram is a rectangle, and that the rectangle formed by the bisectors is then a square.

5. Show that the diagonals of the rectangle are parallel to the sides of the given parallelogram.

6. Show that the diagonals of the rectangle, produced if necessary, bisect the sides of the parallelogram.

7. Show that each diagonal of the rectangle is equal in length to the difference between the lengths of the sides of the given parallelogram.

8. If the lengths of the sides of the parallelogram be a, b , show that the ratio of the area of the rectangle to that of the parallelogram is $(a-b)^2 : 2ab$, and is therefore independent of the shape of the parallelogram. Hence, find the ratio of $a : b$ when the area of the rectangle is equal to that of the parallelogram.

37.

THE POLYNOMIAL THEOREM.

1. Find the general term in the expansion of

$$(a + b + c + \dots + k)^n.$$

2. Find the coefficients of $a_1 a_2^2$ and $a_1 a_2 a_3$ in the expansion of $(a_1 + a_2 + \dots + a_n)^3$, and hence show that

$$(a_1 + a_2 + \dots + a_n)^3 = 3 \sum a_1 \sum a_1^2 - 2 \sum a_1^3 + 6 \sum a_1 a_2 a_3.$$

From the identity

$$(a + b + c)^3 = 3 \sum a \sum a^2 - 2 \sum a^3 + 6abc,$$

deduce that, if $a + b + c = 0$,

$$a^3 + b^3 + c^3 = 3abc.$$

3. Eliminate x, y, z from the equations

$$x + y + z = a, \quad x^2 + y^2 + z^2 = b^2, \quad x^3 + y^3 + z^3 = c^3 \quad \text{and} \quad xyz = d^3.$$

4. Prove that

$a(x^3 + y^3 + z^3) + b(x^2y + x^2z + y^2z + y^2x + z^2x + z^2y) + cxyz$
is divisible by $x + y + z$, if $3a + c = 3b$.

5. Solve the equations

$$x + y + z = 3, \quad x^2 + y^2 + z^2 = 21, \quad x^3 + y^3 + z^3 = 57.$$

6. Show that the sum of the products of the first n integers three together is

$$\frac{1}{4}n(n-2)(n-1)n^2(n+1)^2.$$

7. Find the sum of the squares, and the sum of the cubes, of the roots of the equation

$$ax^n + bx^{n-1} + cx^{n-2} + dx^{n-3} + \dots + k = 0.$$

8. Find the equation, the sums of the first, second and third powers of the roots of which are r, s and t .

38.

APPLICATION OF THE BINOMIAL THEOREM TO THE SOLUTION OF PROBLEMS ON COMBINATIONS.

1. Find the number of homogeneous products, each of r dimensions, that can be formed out of the n letters $a, b, c, \dots$ and their powers.

2. Explain how this theorem may be adapted to certain cases of problems on combinations, for example, to find the number of ways in which five numbers, which may be either 1, 2 or 3, may be written down so that their sum may be 10.

3. If the faces of a die contain respectively the figures 0, 1, 2, 3, 4, 5, in how many ways can a total of 12 be thrown with four such dice ?

40 SUBJECTS FOR MATHEMATICAL ESSAYS

4. If the faces of a die contain respectively the figures 1, 2, 3, 4, 5, 6, in how many ways can a total of 12 be thrown with four such dice ?

5. In how many ways may five numbers, each of which is 3, 4 or 5, be written down so that their sum may be 20 ?

6. In how many ways can 30 balls be taken from a bag which contains 20 red, 15 yellow and 10 blue balls, so that not less than 8 shall be red, 5 yellow and 3 blue ?

7. In an examination, n marks are allotted to each of three papers, and $2n$ to the fourth ; prove that the number of ways in which a candidate may obtain $3n$ marks is

$$\frac{1}{6}(n+1)(5n^2+10n+6).$$

8. In a certain examination, there are four papers set, the maximum number of marks for each paper is 100 ; but, if less than 25 marks be obtained on the first paper, they are not counted towards the aggregate ; find the number of ways in which a candidate may obtain 100 marks.

39.

THE ISOSCELES TRIANGLE WITH EACH ANGLE AT THE BASE DOUBLE OF THE THIRD ANGLE.

1. AB is a straight line divided at D in medial section, a circle is described with A as centre and AB as radius ; BC is a chord of this circle equal to AD ; show that (i) BC touches the circle ACD , (ii) $AD=CD=BC$, and (iii) ABC is an isosceles triangle in which each angle at the base is double of the third angle.

2. Show that ADC (figure of No. 1) is an isosceles triangle, in which one angle is three times each of the others.

3. If $ABCDE$ be a regular pentagon, show that the triangle ACD is an isosceles triangle in which each of the angles C, D is double of the third angle A .

4. If, in the figure of No. 1, CD produced meet the circle with centre A and radius AB in F , show that the triangles FAD, ABC are congruent.

5. Show also that BC, BF are the sides of a regular decagon and a regular pentagon inscribed in that circle.

6. If the bisectors of the angles B, C meet the circle ABC in D, E , show that $AEBCD$ is a regular pentagon.

7. If the circles BCF, ACD (figure of No. 4) intersect again in E , show that CE is equal to BC , and hence that, if G be the mid-point of the minor arc AE , the figure $ADCEG$ is a regular pentagon.

8. If each of the angles B, C of the triangle ABC be double of the angle A , show that

$$\cos \frac{B+C}{2} \cos \frac{A+B+C}{5} = \cos^4 \frac{A+B+C}{4}.$$

40.

TESTS OF CONCURRENCY.

1. $ABCD$ is a quadrilateral; show that the straight lines which join the mid-points of AB and CD , AD and BC , and AC and BD are concurrent and bisect one another.

2. X, Y, Z are points on the sides BC, CA, AB of the triangle ABC , such that

$$AZ^2 + BX^2 + CY^2 = ZB^2 + XC^2 + YA^2;$$

prove that the perpendiculars drawn through X, Y, Z to the corresponding sides are concurrent.

3. AB and PQ are two parallel straight lines, and C, R their mid-points; prove that AP, BQ, CR are concurrent.

4. If two similar polygons be placed so as to have their corresponding sides parallel, the straight lines joining corresponding vertices are concurrent.

5. Show that the perpendiculars from the angular points of a triangle on the sides of the pedal triangle are concurrent.

6. If through the vertices of a triangle ABC , straight lines be drawn through a point O to meet the opposite sides BC , CA , AB in D , E , F respectively, prove that

$$\frac{BD}{DC} \cdot \frac{CE}{EA} \cdot \frac{AF}{FB} = 1; \quad (\text{Ceva's theorem})$$

and, conversely, if D , E , F be points in the sides BC , CA , AB of the triangle ABC , such that

$$\frac{BD}{DC} \cdot \frac{CE}{EA} \cdot \frac{AF}{FB} = 1,$$

then AD , BE , CF are concurrent.

7. D , E , F are points in the sides BC , CA , AB respectively of a triangle ABC , so that AD , BE , CF are concurrent; the circle DEF cuts the sides again in D' , E' , F' ; prove that AD' , BE' , CF' are concurrent.

8. Through a given point F in the diagonal BD of a square $ABCD$, lines are drawn parallel to the sides to meet AB in G , BC in E , CD in K and DA in H ; prove that BH , CF and DG are concurrent.

41.

TESTS OF COLLINEARITY.

1. If OA , OB be straight lines on opposite sides of the straight line COD , so that $\angle AOC = \angle BOD$, show that the points A , O , B are collinear.

Hence, show that two excentres and the intermediate angular point of a triangle are collinear.

2. ABC is a straight line and O any point outside it ; if P, Q, R be the mid-points of OA, OB, OC , show that P, Q, R are collinear.

$OCBA, OEEFG$ are two straight lines inclined at any angle ; if the points P, Q, R of the completed parallelograms $OAPE, OBQF, OCRG$ be collinear, show that the mid-points of AE, BF, CG are collinear.

3. $OACB, OMPN$ are two parallelograms in which OM lies along OA and ON along OB ; if $OM : MP = OA : AC$, show that O, P, C are collinear.

4. If a transversal cut the sides BC, CA, AB of a triangle ABC in D, E, F respectively, show that

$$\frac{BD}{DC} \cdot \frac{CE}{EA} \cdot \frac{AF}{FB} = 1 ; \text{ (Menelaus' theorem)}$$

and conversely, if D, E, F be points on the sides BC, CA, AB respectively of the triangle ABC , so that

$$\frac{BD}{DC} \cdot \frac{CE}{EA} \cdot \frac{AF}{FB} = 1,$$

show that the points D, E, F are collinear.

5. ABC is a triangle, D the mid-point of BC and G a point so that the triangles AGB, AGC are equivalent ; prove that A, G, D are collinear.

Hence, prove that the medians of a triangle are concurrent.

6. By means of the converse of Menelaus' theorem, show that the following sets of points are collinear :

(i) The points in which the three external bisectors of the angles of a triangle meet the opposite sides produced.

(ii) The points in which two internal and one external bisector of the angles of a triangle meet the opposite sides, in the latter case produced.

7. If two triangles be placed so that the straight lines joining corresponding angular points in the two triangles are concurrent, prove that their corresponding sides intersect in points which are collinear (Desargues' theorem).

8. $ABCD$ is a quadrilateral ; AB, DC produced meet in E , and BC, AD produced in F ; the straight lines which bisect the angles A, C internally meet in X , those which bisect the angles B, D internally in Y , and those which bisect the angles E, F externally in Z ; by considering a system of four equal forces acting respectively along AB, AD, CB, CD , show that the points X, Y, Z are collinear.

42.

COMMON TANGENTS AND CENTRES OF SIMILITUDE.

1. Draw the four common tangents to two circles which do not intersect.

2. The radii of two circles are a, b ($a > b$), and the distance between their centres is c ($> a + b$) ; find the length (i) of an external, (ii) of an internal, common tangent.

Example, $a=5, b=3, c=12$ ins.

3. If (i) an incircle and an excircle, or (ii) two excircles, of a triangle be given, construct the triangle.

4. The external common tangents and the internal common tangents of two circles respectively intersect on the line through the centres of the circles. (The points of intersection are called the *external* and *internal centres of similitude* respectively.)

5. A common tangent to two circles meets the line of centres OO' in S ; and a straight line $SP'Q'PQ$ meets the circle with centre O' in P', Q' and the circle with centre O in P, Q ; show that OP is parallel to $O'P'$ and OQ to $O'Q'$.

6. Also show that

$$\text{rect. } SP \cdot SQ' = \text{rect. } SP' \cdot SQ = \text{rect. } SA \cdot SB.$$

7. Two circles intersect in A and their common tangents in O ; OA meets the circles in B, C ; prove that OA is a mean proportional between OB, OC .

8. If there be three circles, the centres of which are not collinear, show that (i) the straight lines joining the centre of each circle with the internal centre of similitude of the other two are concurrent; (ii) the external centre of similitude of any pair of circles and the internal centres of similitude of the other two pairs are collinear; and (iii) the three external centres of similitude are collinear.

43.

RESOLUTION OF THE GENERAL EXPRESSION OF THE SECOND DEGREE IN x AND y INTO LINEAR FACTORS.

1. If the general expression of the second degree in x, y ,

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c,$$

be the product of two expressions of the first degree,

$$lx + my + n \quad \text{and} \quad l'x + m'y + n',$$

prove that

$$a = ll', \quad b = mm', \quad c = nn', \quad 2f = mn' + m'n,$$

$$2g = nl' + n'l, \quad 2h = lm' + l'm.$$

2. Hence, show that the expression

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c$$

can be resolved into two factors of the first degree if

$$2fgh = af^2 + bg^2 + ch^2 - abc.$$

3. If a be not zero, show that the same condition may be obtained from the condition that the quadratic in x ,

$$ax^2 + 2x(hy + g) + by^2 + 2fy + c = 0,$$

may have rational roots.

4. Find the condition that the expression

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c$$

may be an exact square.

5. If $2hxy + 2gx + 2fy + c$ can be resolved into two factors of the form $ax + \beta$ and $\gamma x + \delta$, show that $2fg = ch$.

6. Resolve the following expressions into factors of the first degree in x and y :

$$(i) \ x^2 + xy - 6y^2 + 5x + 6,$$

$$(ii) \ x^2 - 6xy + 9y^2 + 4x - 12y + 4,$$

$$(iii) \ 12xy - 8x + 9y - 6.$$

7. Find λ in order that

$$(i) \ 9x^2 - 3xy + \lambda y^2 + 15x - 16y - 14 = 0$$

may be the product of two expressions of the first degree in x and y ;

$$(ii) \ 4x^2 - 10xy + 10y^2 + \lambda(3x^2 - 10xy + 3y^2)$$

may be a complete square.

8. If the general equation of the second degree,

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0,$$

represent two straight lines, find (i) the condition for perpendicularity, and (ii) the coordinates of their point of intersection.

44.

PAIRS OF STRAIGHT LINES THROUGH THE ORIGIN.

1. Show that the equation $ax^2 + 2hxy + by^2 = 0$ represents a pair of straight lines through the origin ; find the angle

between them, and show that the lines are real and different, real and coincident, or imaginary and different, according as h^2 is greater than, equal to, or less than, ab ; also show that the lines are at right angles if $a + b = 0$.

2. Find the angle between the lines $x^2 + xy - 6y^2 = 0$.

3. Find the condition that the lines $ax^2 + 2hxy + by^2 = 0$ may be equally inclined to the axis of x .

4. Show that the equation of the lines through the origin perpendicular to the lines $ax^2 + 2hxy + by^2 = 0$ is

$$bx^2 - 2hxy + ay^2 = 0.$$

5. Show that the equation of the lines bisecting the angles between the lines $ax^2 + 2hxy + by^2 = 0$ is

$$hx^2 - (a - b)xy - hy^2 = 0.$$

6. Find the area of the triangle contained by the lines

$$ax^2 + 2hxy + by^2 = 0 \quad \text{and} \quad lx + my = n.$$

7. Find the equation of the tangents drawn from the origin to the circle $x^2 + y^2 + 2gx + 2fy + c = 0$.

8. Find the condition that the lines joining the origin to the points of intersection of the line $lx + my = n$ and the curve $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ may be at right angles.

45.

EQUATION OF THE STRAIGHT LINES JOINING THE ORIGIN TO THE POINTS OF INTERSECTION OF A STRAIGHT LINE AND A CONIC.

1. Find the equation of the straight lines joining the origin to the points of intersection of the straight line $lx + my = n$ and the conic $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$.

2. If the straight lines joining the origin to the points of intersection of the line $x+2y=3$ and the circle

$$x^2+y^2+4x-6y+c=0$$

be at right angles, find the value of c .

3. Find the equation of the polar of the origin with respect to the circle

$$x^2+y^2+2gx+2fy+c=0;$$

and hence find the equation of the tangents drawn from the origin to this circle.

4. If the lines joining the vertex of a parabola to the ends of a chord be perpendicular, the chord intersects the axis in a fixed point.

5. If the chord of a parabola subtend a right angle at the vertex, find the locus of its pole.

6. A straight line $y=mx+c$ cuts the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ in the points P, Q ; find the value of c if P, Q be the ends of conjugate diameters.

7. A straight line through a fixed point $(0, c)$ on the axis of the parabola $y^2=4ax$ cuts the parabola in P, Q ; if O be the vertex, find the area of the triangle OPQ , and show that it is a minimum when PQ is perpendicular to the axis.

8. A variable chord of an ellipse is such that the lines joining the centre to its ends are always at right angles; show that the locus of its pole is an ellipse, the excentricity of which is to that of the original ellipse as $\sqrt{(a^2+b^2)} : a$.

46.

CONCURRENT CIRCLES.

1. The three circles which pass through the ends of each side of a triangle and the corresponding excentre are concurrent.

2. Find the condition that the circles described on the sides of a quadrilateral as diameters may be concurrent.

3. ABC is a triangle, D, E, F any points on the sides BC, CA, AB respectively; show that the circles AEF, BFD, CDE are concurrent.

4. Equilateral triangles are described on the sides of any triangle outside the triangle; show that their circumcircles are concurrent. Hence, show how to find a point inside a triangle at which the sides subtend equal angles.

5. Three points P, Q, R move so that their distances from pairs of angular points of the triangle ABC are governed by the ratios

$$BP : CP = AB : AC,$$

$$CQ : AQ = BC : BA,$$

$$AR : BR = CA : CB;$$

prove that the circles so traced are concurrent.

6. ABC is a triangle; prove that the circles which touch BC at B , CA at C , AB at A , and pass respectively through A, B, C , are concurrent.

7. If BCP, CAQ, ABR be equilateral triangles described externally on the sides of a triangle ABC , prove that the point of intersection of AP and BQ is concyclic with B, C and P , and hence that AP, BQ, CR are concurrent.

8. The circumcircles of the four triangles formed by four intersecting straight lines are concurrent.

47.

COMPLEX QUANTITY. I.

1. If $\cos \alpha + i \sin \alpha$, where $i = \sqrt{-1}$, be denoted by $(1, \alpha)$, show that $(1, \alpha) \times (1, \beta) = (1, \alpha + \beta)$ and $(1, \alpha) \times (1, \alpha) = (1, 2\alpha)$, and, by induction, that $(1, \alpha)^n = (1, n\alpha)$.

2. Show that $(1, n\alpha) = \left(1, n \cdot \alpha + \frac{2r\pi}{n}\right)$,

and hence that $\left(1, \alpha + \frac{2r\pi}{n}\right)$ is an n th root of $(1, n\alpha)$.

3. Show that all the n th roots of $(1, n\alpha)$ are obtained by giving to r the values $0, 1, 2, \dots, (n-1)$ in the expression $\left(1, \alpha + \frac{2r\pi}{n}\right)$. Also, show that, if any other integral value be given to r , one of the above values is repeated.

4. Find all the cube roots of 1 and -1 , and the fourth roots of $\sqrt{-1}$ and $-\sqrt{-1}$.

5. Find the fourth roots of $1 + \sqrt{-3}$ and the cube roots of $\sqrt{3} + i$.

6. From the equation

$$(1, \alpha) \times (1, \beta) \times (1, \gamma) \times \dots = (1, \alpha + \beta + \gamma + \dots),$$

deduce (i) the formulae for

$$\cos(\alpha + \beta) \quad \text{and} \quad \sin(\alpha + \beta),$$

and (ii) the formulae for

$$\cos(\alpha_1 + \alpha_2 + \dots + \alpha_n) \quad \text{and} \quad \sin(\alpha_1 + \alpha_2 + \dots + \alpha_n).$$

7. From the equation

$$\cos n\alpha + i \sin n\alpha = (\cos \alpha + i \sin \alpha)^n,$$

deduce that

$$\begin{aligned} \cos n\alpha &= \cos^n \alpha - {}_nC_2 \cdot \cos^{n-2} \alpha \sin^2 \alpha + {}_nC_4 \cdot \cos^{n-4} \alpha \sin^4 \alpha - \dots, \\ \sin n\alpha &= {}_nC_1 \cdot \cos^{n-1} \alpha \sin \alpha - {}_nC_3 \cdot \cos^{n-3} \alpha \sin^3 \alpha + \dots \end{aligned}$$

8. If $\cos \alpha + \cos \beta + \cos \gamma = 0$ and $\sin \alpha + \sin \beta + \sin \gamma = 0$, prove that

$$\cos 3\alpha + \cos 3\beta + \cos 3\gamma = 3 \cos (\alpha + \beta + \gamma)$$

and
$$\sin 3\alpha + \sin 3\beta + \sin 3\gamma = 3 \sin (\alpha + \beta + \gamma).$$

48.

FACTORS OF THE EXPRESSION $x^3 + y^3 + z^3 - 3xyz$.

1. Prove that

$$x^3 + y^3 + z^3 - 3xyz = (x + y + z) \cdot \frac{1}{2} \{ (y - z)^2 + (z - x)^2 + (x - y)^2 \}.$$

Hence, show that $x^3 + y^3 + z^3 > 3xyz$, and therefore that

$$\frac{1}{3}(a + b + c) > \sqrt[3]{(abc)}.$$

2. Prove that

$$(b - c)^3 + (c - a)^3 + (a - b)^3 = 3(b - c)(c - a)(a - b).$$

3. If ω, ω^2 be the two complex cube roots of unity, show that

$$a^3 + b^3 + c^3 - 3abc = (a + b + c)(a + b\omega + c\omega^2)(a + b\omega^2 + c\omega).$$

4. If $X = ax + cy + bz$, $Y = cx + by + az$, $Z = bx + ay + cz$, show that

$$(a^3 + b^3 + c^3 - 3abc)(x^3 + y^3 + z^3 - 3xyz) = X^3 + Y^3 + Z^3 - 3XYZ.$$

5. Show that the sum of the cubes of the roots of the equation $x^3 + 3qx - r = 0$ is $3r$.

6. Prove that

$$(x^2 + 2yz)^3 + (y^2 + 2zx)^3 + (z^2 + 2xy)^3 - 3(x^2 + 2yz)(y^2 + 2zx)(z^2 + 2xy) = (x^3 + y^3 + z^3 - 3xyz)^2.$$

7. If $\cos \alpha + \cos \beta + \cos \gamma = 0$, show that

$$\cos 3\alpha + \cos 3\beta + \cos 3\gamma = 12 \cos \alpha \cos \beta \cos \gamma.$$

8. If $\cos \alpha + \cos \beta + \cos \gamma = 0$ and $\sin \alpha + \sin \beta + \sin \gamma = 0$, prove that

$$\cos 3\alpha + \cos 3\beta + \cos 3\gamma = 3 \cos (\alpha + \beta + \gamma)$$

and
$$\sin 3\alpha + \sin 3\beta + \sin 3\gamma = 3 \sin (\alpha + \beta + \gamma).$$

49.

CORRESPONDING ALGEBRAICAL AND TRIGONOMETRICAL IDENTITIES.

1. Prove that

$\tan 3\alpha \tan 2\alpha \tan \alpha = \tan 3\alpha - \tan 2\alpha - \tan \alpha$,
and deduce the corresponding algebraical formula when $\tan \alpha = a$.

2. If $yz + zx + xy = 1$, show that

$$2(x^2 + y^2 + z^2 - 1) = (y - z)^2 + (z - x)^2 + (x - y)^2.$$

Deduce the corresponding trigonometrical identity if $\tan A$ be put for x , etc., where $A + B + C = \frac{\pi}{2}$, and show that

$$\tan^2 A + \tan^2 B + \tan^2 C \leq 1.$$

3. If $yz + zx + xy = 1$, prove that

$$\frac{x}{1-x^2} + \frac{y}{1-y^2} + \frac{z}{1-z^2} = \frac{4xyz}{(1-x^2)(1-y^2)(1-z^2)}.$$

Also obtain the corresponding trigonometrical formula by putting $\tan \frac{A}{2}$ for x , etc.

4. Show, by the aid of trigonometry, that, if $x + y + z = xyz$,

$$\frac{3x - x^3}{1 - 3x^2} + \frac{3y - y^3}{1 - 3y^2} + \frac{3z - z^3}{1 - 3z^2} = \frac{(3x - x^2)(3y - y^2)(3z - z^2)}{(1 - 3x^2)(1 - 3y^2)(1 - 3z^2)}.$$

5. If $a + b + c = 0$, show, by substituting $(1, a)$ for a , etc., in the formula $a^3 + b^3 + c^3 = 3abc$, that

$$\cos 3\alpha + \cos 3\beta + \cos 3\gamma = 3 \cos (\alpha + \beta + \gamma)$$

and $\sin 3\alpha + \sin 3\beta + \sin 3\gamma = 3 \sin (\alpha + \beta + \gamma).$

6. In the identity $a^3 - b^3 = (a - b)(a^2 + ab + b)$, put $a = (1, \alpha)$, $b = (1, \beta)$, and deduce that

$$\begin{aligned} \cos 3\alpha - \cos 3\beta &= (\cos \alpha - \cos \beta)(\cos 2\alpha + \cos \overline{\alpha + \beta} + \cos 2\beta) \\ &\quad - (\sin \alpha - \sin \beta)(\sin 2\alpha + \sin \overline{\alpha + \beta} + \sin 2\beta), \end{aligned}$$

with a similar formula for $\sin 3\alpha - \sin 3\beta$.

7. If $a+b+c=0$, prove that

$$a^3(b-c) + b^3(c-a) + c^3(a-b) = 0,$$

and hence show, by substituting $(1, \alpha)$ for a , etc., that

$$\begin{aligned} \sin 2(\beta + \gamma) \sin (\beta - \gamma) + \sin 2(\gamma + \alpha) \sin (\gamma - \alpha) \\ + \sin 2(\alpha + \beta) \sin (\alpha - \beta) = 0. \end{aligned}$$

8. Show that

$$\frac{3}{1+x^3} = \frac{1}{1+x} + \frac{1}{1-\alpha x} + \frac{1}{1-\beta x},$$

where α, β are the complex cube roots of -1 , and deduce, by putting $x = (1, 2\theta)$, that

$$3 \tan 3\theta = \tan \theta - \cot \left(\theta + \frac{\pi}{6} \right) - \cot \left(\theta - \frac{\pi}{6} \right).$$

50.

ORTHOGONAL CIRCLES.

1. If two circles cut orthogonally, the square on the line joining their centres is equal to the sum of the squares on the radii, and conversely.

Hence, show that only one of a series of concentric circles can be cut orthogonally by a given circle.

2. Draw a circle of given radius so as to cut orthogonally a given circle.

3. If two circles cut orthogonally, the rectangle contained by their common chord and their line of centres is equal to twice the rectangle contained by their radii.

4. If the circumcircle of a triangle ABC and the excircle opposite A cut orthogonally, prove that the radius of the excircle is equal to the diameter of the circumcircle.

5. ABC is a triangle; show that the radii of the circles

with centres A, B, C , each of which cuts the other two orthogonally, are $\sqrt{(bc \cos A)}$, $\sqrt{(ca \cos B)}$ and $\sqrt{(ab \cos C)}$.

6. If O be the orthocentre of the triangle ABC , prove that the circles on AB, OC as diameters are orthogonal.

7. Show that every circle with its centre on the radical axis of two given circles and radius equal to the tangents from that point cuts the given circles orthogonally.

Hence, show how to draw a circle so as to cut three given circles (the centres of which are not collinear) orthogonally.

8. If two circles cut one another at A, B , and if each cut a third circle orthogonally, prove that any circle drawn through A, B will cut the third circle orthogonally.

51.

DIP OF A STRATUM.

1. A plane is inclined at an angle α to the south ; a road is made up the plane in the direction ϕ E. of N. ; if δ be the inclination of the road to the horizon, prove that

$$\tan \delta = \tan \alpha \cos \phi.$$

2. The line of greatest slope of a plane is inclined at an angle α to the horizon ; a line on the inclined plane, making an angle θ with the line of greatest slope, is inclined at an angle δ to the horizon ; show that

$$\sin \delta = \sin \alpha \cos \theta.$$

3. The inclinations of a plane to the horizon are α, β along lines whose horizontal directions are inclined at an angle λ ; if the greatest inclination of the plane be δ in a horizontal direction inclined at an angle θ to that of the inclination α , prove that

$$\tan^2 \delta = (\tan^2 \alpha + \tan^2 \beta - 2 \tan \alpha \tan \beta \cos \lambda) \operatorname{cosec}^2 \lambda,$$

$$\tan \theta = (\tan \beta - \tan \alpha \cos \lambda) / \tan \alpha \sin \lambda.$$

What do these formulae become when λ is a right angle ?

4. Find the amount and direction of the true dip of a stratum, the apparent dips of which in the directions N. 40° E. and S. 50° E. are respectively 32° and 18° .

5. At A , B , C , three points on horizontal ground, vertical bores are made and meet a stratum of coal at depths x , y , z respectively ; find the amount and direction of the dip of the stratum in terms of x , y , z and the elements of the triangle ABC .

6. A vertical post of height h rises from a plane inclined towards the south at an angle α ; find the length of its shadow, (i) when the sun is due south at an elevation β , (ii) when the sun is θ west of south at an elevation β .

7. Two planes are inclined to the horizon at angles α , β in directions which make an angle λ with one another ; find the direction and inclination to the horizon of their common section.

8. Show that the inclination δ to the horizon of the common section of two planes inclined at the same angle to the horizon is given by

$$\tan \delta = \tan \alpha \cos \frac{\lambda}{2},$$

where λ is the horizontal angle between the lines of greatest slope.

52.

NON-PLANAR LINES.

1. A straight line can be drawn perpendicular to each of two non-planar lines, and this line is the shortest distance between the two given lines.

2. If a straight line be parallel to a plane, the shortest distance between it and any line in the plane is constant.

3. If AB be the common normal to two non-planar lines and CD any other line joining them, show that the line joining the mid-points of AB , CD is perpendicular to AB .

4. The shortest distance between two equal non-planar lines AB , CD , which are at right angles, bisects each of them and is equal to $AB/\sqrt{2}$; show that AC , AB , AD are all equal.

5. Through a given point, draw a straight line so as to intersect two non-planar lines. Show that only one such line can be drawn.

6. Through a given point, draw a straight line which shall make equal angles with each of three non-planar lines.

7. $ABCD$, $CDEF$ are faces of a cube and G is the mid-point of the edge EF ; draw the straight line which is perpendicular to AG and to the edge BH .

8. a , b , c are three non-intersecting straight lines, of which a and b are perpendicular to c ; x , y , z are respectively the shortest distances between b and c , c and a , a and b ; prove that z is equal to the shortest distance between x and y .

53.

THE CUBE.

1. The square on the diagonal of a cube is equal to three times the square on one of its edges.

2. Every section of a cube by a plane parallel to one face is a square.

3. Every section of a cube by a plane through the three angular points adjacent to a given angle is an equilateral triangle.

4. Show how a plane must pass so that its common section with a cube may be a pentagon, and show that this pentagon can never be regular.

5. Show how to draw a plane so that its section with a cube may be a regular hexagon.

6. Show that a cube may be cut by a plane so that the section is a square, the area of which is to that of one face of the cube as 9 to 8.

7. The shortest distance between a diagonal of a cube and an edge which it does not meet is one-half the diagonal of one of the faces.

8. Each corner of a cube is cut off by a plane passing through the mid-points of the three edges which meet in that corner ; state the number and forms of the faces in the resulting polyhedron, the number of vertices and the number of edges, and compare the surface and volume of the resulting polyhedron with those of the original cube.

54.

THE REGULAR OCTAHEDRON.

1. Prove that it is possible to form a solid angle the faces of which are four equilateral triangles.

2. Show that the regular polyhedron in which each solid angle is contained by four equilateral triangles has 6 vertices, 12 edges and 8 faces.

3. Find the distance between two opposite corners of a regular octahedron of edge a .

4. Find the surface and volume of a regular octahedron of edge a .

5. Find the radii of the inscribed and circumscribed spheres of a regular octahedron of edge a .

6. Find the cosine of the dihedral angle between two adjoining faces of a regular octahedron.

7. A regular tetrahedron and a regular octahedron have the same surface ; compare their volumes.

8. If the six corners of a regular octahedron be cut off by planes passing through the mid-points of the edges meeting in each corner, find the number of vertices, edges and faces in the resulting polyhedron. Find also the shapes of the faces, and compare the total surface and volume of the new polyhedron with those of the given octahedron.

55.

RANGE OF PROJECTILES ON AN INCLINED PLANE.

1. From a point in a plane inclined at an angle β to the horizon, a particle is projected in a vertical plane through a line of greatest slope ; the initial velocity is u and the direction of projection makes an angle α with the horizon ; find the resolved parts of its acceleration, and of its velocity at any subsequent time, in directions parallel and perpendicular to the plane.

2. Find (i) the time of flight directly up and down the plane, (ii) the range directly up the plane, and (iii) the range directly down the plane.

3. A particle is projected at an angle α to the horizon from the foot of a plane of inclination β ; show that it will strike the plane at right angles if

$$\cot \beta = 2 \tan (\alpha - \beta).$$

4. Find the maximum range directly up and down the plane, and show that the direction of projection in either case bisects the angle between the line of greatest slope and the line drawn vertically upwards from the point of projection.

5. Show that, corresponding to a given range on an inclined plane, there are two directions of projection, which are equally inclined to the direction which gives the maximum range.

6. Show that the greatest range on an inclined plane through the point of projection is equal to the distance through which the particle would fall freely in its time of flight.

7. A vertical plane through the point of projection cuts the inclined plane along a line which makes an angle θ with the line of greatest slope ; find (i) the inclination of this line to the horizon, and (ii) the maximum range up and down this line.

8. Hence, show that the area commanded by a gun on an inclined plane is an ellipse with a focus at the given point and its major axis along the line of greatest slope.

56.

THE EQUATION OF THE TANGENT TO A PARABOLA

IN THE FORM $y = mx + \frac{a}{m}$.

1. If the line $y = mx + c$ touch the parabola $y^2 = 4ax$, show that $c = a/m$.

2. Comparing the two equations of the tangent,

$$y = mx + \frac{a}{m} \quad \text{and} \quad yy' = 2a(x + x'),$$

show that the point of contact of the former is

$$(a/m^2, 2a/m).$$

3. Find the point of intersection of the tangents

$$y = mx + \frac{a}{m}, \quad y = m'x + \frac{a}{m'}$$

to a parabola, and show that, if the tangents be at right angles, the locus of their point of intersection is the directrix.

4. Find the equation of the chord of contact of the tangents

$$y = mx + \frac{a}{m}, \quad y = m'x + \frac{a}{m'}.$$

5. Find the locus of the foot of the perpendicular from the focus of the parabola $y^2 = 4ax$ on the tangent

$$y = mx + \frac{a}{m}.$$

6. Show that the orthocentre of the triangle formed by three tangents to a parabola lies on the directrix.

7. Three tangents are drawn to a parabola; show that the circumcircle of the triangle formed by them passes through the focus.

8. If the equations of two tangents to a parabola be

$$y = mx + \frac{a}{m}, \quad y = m'x + \frac{a}{m'},$$

find the point of intersection of the corresponding normals. Hence, show that, if tangents be drawn to a parabola $y^2 = 4ax$ from points on the parabola $y^2 = a(x+a)$, the point of intersection of corresponding pairs of normals lies on a straight line perpendicular to the common axis of the parabolas.

57.

THE NORMALS TO A PARABOLA.

1. Find the equation of the normal to the parabola $y^2 = 4ax$ at the point $(am^2, -2am)$.

2. Through any point, it is possible to draw three normals (of which one at least must be real) to a parabola.

3. If the normals at three points be concurrent, show that the algebraical sum of the ordinates is zero.

4. Normals to the parabola $y^2=4ax$ at the points P, Q, R meet in the point (h, k) ; find the coordinates of the centroid of the triangle PQR .

5. If two of the normals to the parabola $y^2=4ax$ from a point P be at right angles, prove that the locus of P is

$$y^2=a(x-3a).$$

6. If P, Q, R be the feet of the normals which pass through a given point, show that the circle PQR passes through the vertex.

7. If, from points P, Q , on the same side of the axis of a parabola, normals PC, QC be drawn making complementary angles with the axis, and UCN the third normal through C meet the curve in U and the axis in N , then UN is bisected at C .

8. From any point in the normal to a parabola, two other normals are drawn to the curve; prove that the straight line joining their feet is parallel to a fixed straight line.

58.

EULER'S POLYHEDRON THEOREM.

1. If a cube be constructed, face by face, state the number of edges and of vertices in the partially constructed figure for each value of F from 1 to 6.

2. If V be the number of vertices, E the number of edges, and F the number of faces in any polyhedron, prove Euler's theorem, that

$$E+2=F+V.$$

3. In the pentagonal dodecahedron, in which each solid angle is contained by the plane angles of three equal regular pentagons, show that the number of vertices is 20, of edges 30, and of faces 12.

4. A polyhedron is contained by 8 triangles and 11 quadrilaterals; find the number of vertices.

5. No polyhedron can have less than six edges, and, with the exception of six edges, none can have less than eight edges.

6. If the faces of a polyhedron be all triangular, the number of faces is $2V - 4$, where V is the number of vertices.

7. Prove that the sum of all the plane angles of the faces of a polyhedron with V vertices is $4V - 8$ right angles or double the sum of the interior angles of a convex polygon with the same number of vertices.

8. A polyhedron with F faces has all its faces triangular; prove that the number of vertices is $\frac{1}{2}(F + 4)$.

If $F = 10$, prove that there is at least one vertex in which more than four faces meet.

59.

THE TETRAHEDRON AND ITS CIRCUMSCRIBING PARALLELEPIPED.

1. Through two non-planar lines, one, and only one, pair of parallel planes can be drawn.

2. Defining each pair of edges of a tetrahedon which do not meet as *opposite edges*, show that the three pairs of parallel planes drawn through opposite edges form a parallelepiped, and that each edge of the tetrahedron is a diagonal of a face of the parallelepiped.

3. If two opposite edges of a tetrahedron be equal, the corresponding faces of the circumscribing parallelepiped are rectangles; and, if two opposite edges of a tetrahedron be at right angles, the corresponding faces of the parallelepiped are rhombuses.

4. If two pairs of opposite edges of a tetrahedron be mutually at right angles, the third pair are also at right angles.

5. If each edge of a tetrahedron be equal to the opposite edge, show that the straight lines which join the mid-points of opposite edges are mutually perpendicular.

6. One edge AD of the tetrahedron $ABCD$ is perpendicular to the edge BC ; prove that the sum of the squares on AB , CD is equal to the sum of the squares on AC , BD .

7. The volume of a tetrahedron is one-third that of its circumscribing parallelepiped.

8. Hence, show that all tetrahedra are of equal volume in which a pair of opposite edges remains of the same length and in the same straight line.

60.

PLANE SECTIONS OF A TETRAHEDRON.

1. If a straight line be parallel to a plane, the common section of the plane with any plane passing through the given line is parallel to that line.

2. Show that it is possible to draw a plane parallel to each of two non-planar lines.

3. If a tetrahedron be cut by a plane parallel to a pair of opposite edges, show that the common section of the tetrahedron by the plane is a parallelogram.

4. If the opposite edges to which the plane is parallel be equal, show that the perimeter of the parallelogram is constant. Also, show that this is not the case if the opposite edges be unequal.

5. If a plane parallel to the edges AB , CD of a tetrahedron meet the edge BC in P , show that the area of the parallelogram of intersection varies as $BP \cdot PC$.

6. Show that the area of the parallelogram is a maximum when the plane bisects all four edges to which it is not parallel.

7. If a plane section of a tetrahedron be a rhombus, the side of the rhombus is half the harmonic mean between a pair of opposite edges.

8. Show that the centre of the common section of a tetrahedron and a plane parallel to a pair of opposite edges lies on the straight line joining the mid-points of those opposite edges.

Show that the centre of gravity of a tetrahedron lies on the straight line joining the mid-points of a pair of opposite edges, and, hence, that the three lines joining the mid-points of opposite edges are concurrent.

61.

SUM OF n TERMS OF THE SERIES

$$1.2.3 + 2.3.4 + 3.4.5 + \dots$$

Find the sum of n terms of the series

$$1.2.3 + 2.3.4 + 3.4.5 + \dots$$

1. By means of the auxiliary series

$$1.2.3.4 + 2.3.4.5 + 3.4.5.6 + \dots$$

2. By induction.

3. By summing the series $\Sigma(r^3 + 3r^2 + 2r)$ from $r=1$ to $r=n$.

4. If $n(n+1)(n+2) = an^4 + bn^3 + cn^2 + dn$
 $- \{a(n-1)^4 + b(n-1)^3 + c(n-1)^2 + d(n-1)\}$,

find, by giving different values to n , the values of a, b, c, d .

5. Hence find the sum of the given series.

6. Find the sum of the coefficients of the first n terms in the expansion of $(1-x)^{-4}$, and deduce the sum of the given series.

7. Extend the method of either No. 1 or No. 2 so as to find the sum of n terms of the series of which the s th term is

$$s(s+1)(s+2) \dots (s+r-1).$$

8. Extend the method of No. 6 so as to find the sum of the series in No. 7.

62.

CONSTRUCTION OF LOGARITHMIC TABLES.

1. Given that

$$\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \dots - (-1)^n \frac{x^n}{n} + \dots,$$

prove that

$$\log \frac{1+x}{1-x} = 2 \left(x + \frac{x^3}{3} + \frac{x^5}{5} + \dots \right),$$

and hence, putting $x = \frac{m-n}{m+n}$, that

$$\log \frac{m}{n} = 2 \left[\frac{m-n}{m+n} + \frac{1}{3} \left(\frac{m-n}{m+n} \right)^3 + \frac{1}{5} \left(\frac{m-n}{m+n} \right)^5 + \dots \right].$$

2. Putting $m=2, n=1$, find $\log 2$ to 4 places of decimals.

3. If the terms neglected in finding $\log 2$ to 4 places of decimals be those beginning with $\frac{1}{11} \cdot \frac{1}{3^{11}}$, show that the sum of the terms neglected is less than

$$\frac{1}{11} \left(\frac{1}{3^{11}} + \frac{1}{3^{13}} + \frac{1}{3^{15}} + \dots \right),$$

and therefore cannot affect even the fifth place of decimals.

4. From the second series in No. 1, find $\log 3$ to 4 places of decimals.

5. Prove that

$$\log_e m = \log_a m \times \log_e a = \log_a m / \log_a e.$$

6. Assuming that $1/\log_e 10 = .43429\dots$, find $\log_{10} 2$ and $\log_{10} 3$ to four places of decimals.

7. Show that the difference between $\log n$ and $\log (n+1)$ decreases as n increases.

8. Prove that the differences in the tables of common logarithms in the neighbourhood of the numbers 9876 and 12345 are roughly as 5 : 4.

63.

THE EXCENTRIC ANGLE OF A POINT ON AN ELLIPSE.

1. Define the excentric angle of a point on an ellipse, and find the coordinates of the point in terms of the excentric angle.

2. Show that the difference between the excentric angles of the ends of two conjugate diameters of an ellipse is a right angle. Hence, show that the sum of the squares of two conjugate semi-diameters is constant.

3. If ϕ , ϕ' be the excentric angles of two points on an ellipse, find the equation of the secant through the points; and deduce the equation of the tangent at the point.

4. Find the coordinates of the point of intersection of tangents at two points on an ellipse of which the excentric angles are given.

5. Find the equation of the normal at a point with excentric angle ϕ on an ellipse. If the normal make an angle θ with the central radius drawn to the point, prove that

$$2ab \tan \theta = (a^2 - b^2) \sin 2\phi.$$

6. Find the coordinates of the point of intersection of normals to an ellipse at points the excentric angles of which are $\phi + \theta$ and $\phi - \theta$.

7. If ϕ , ϕ' be the excentric angles of the points of contact of two tangents drawn from the point (h, k) to an ellipse, prove that

$$\tan \frac{\phi + \phi'}{2} = \frac{ak}{bh} \quad \text{and} \quad \tan \frac{\phi - \phi'}{2} = \sqrt{\left(\frac{h^2}{a^2} + \frac{k^2}{b^2} - 1\right)}.$$

8. Find the relation between the excentric angles of four concyclic points of an ellipse.

64.

COAXAL CIRCLES.

1. Show that the square on the tangent from the point (x, y) to the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ is

$$x^2 + y^2 + 2gx + 2fy + c.$$

2. Show that the locus of points from which the tangents to the circles

$x^2 + y^2 + 2gx + 2fy + c = 0$ and $x^2 + y^2 + 2g'x + 2f'y + c' = 0$ are equal: (i) is a straight line perpendicular to the line of

centres, and (ii) passes through the points of intersection, real or imaginary, of the given circles.

3. The three radical axes of three circles taken in pairs are concurrent.

4. If the line of centres be taken as the axis of x , and the radical axis of two given circles as the axis of y , show that the equations of the circles may be written in the form

$$x^2 + y^2 + 2gx + c = 0 \quad \text{and} \quad x^2 + y^2 + 2g'x + c = 0.$$

5. If a system of circles be such that every pair has the same radical axis (that is, if the circles be *coaxal*), show that the equations of the circles of zero radius (that is, the *limiting points* of the system) may be written in the form

$$x^2 + y^2 \pm 2\sqrt{c} \cdot x + c = 0.$$

6. If $x^2 + y^2 + 2gx + c = 0$ be the equation of one of a system of coaxal circles, show that the polar of either of the limiting points $(\sqrt{c}, 0)$, $(-\sqrt{c}, 0)$ passes through the other.

7. Show that each circle of a coaxal system is cut orthogonally by any circle that passes through the limiting points of the system.

8. Show that the general equation of circles coaxal with the circles

$$x^2 + y^2 + 2gx + 2fy + c = 0, \quad x^2 + y^2 + 2g'x + 2f'y + c' = 0$$

is $x^2 + y^2 + 2gx + 2fy + c + \lambda(x^2 + y^2 + 2g'x + 2f'y + c') = 0$,
 λ being any constant.

65.

THE PATH OF A PROJECTILE.

1. A particle is projected from a point O with velocity u ft. per sec. in a direction inclined at an angle α to the horizon; if P be the position of the particle after t seconds, and if a

vertical line PM meet the horizontal line through O in M , show that

$$OM = u \cos \alpha \cdot t \quad \text{and} \quad PM = u \sin \alpha \cdot t - \frac{1}{2}gt.^2$$

2. If A be the highest point, and if the vertical line AL meet the horizontal line through O in L , show that the time of reaching A is $u \sin \alpha/g$, and hence that

$$OL = u^2 \cos \alpha \sin \alpha/g \quad \text{and} \quad AL = u^2 \sin^2 \alpha/2g.$$

3. If the horizontal line through P meet the vertical line through A in N , show that

$$PN = u \cos \alpha \left(\frac{u \sin \alpha}{g} - t \right) \quad \text{and} \quad AN = \frac{g}{2} \left(\frac{u \sin \alpha}{g} - t \right)^2.$$

4. Hence, show that

$$PN^2 = \frac{2u^2 \cos^2 \alpha}{g} \cdot AN,$$

that is, that the path of the projectile is a parabola with axis vertical and latus rectum $2u^2 \cos^2 \alpha/g$.

5. Show that the height of the focus above the horizontal line through O is $-u^2 \cos 2\alpha/g$, and thus that, if the range on an inclined plane through the point of projection be a maximum, the focus of the path will lie in the inclined plane.

6. Show that the height of the directrix above the horizontal through O is $u^2/2g$, and thus that the height of the directrix above the horizontal line through P is

$$\frac{1}{2g}(u^2 - 2u \sin \alpha \cdot gt + g^2t^2).$$

7. Show that the velocity of the particle at P is

$$\sqrt{(u^2 - 2u \sin \alpha \cdot gt + g^2t^2)}.$$

8. Hence, show that the velocity of the particle at P is equal in magnitude to that which would be attained by falling from the directrix to P .

66.

MOMENTS OF FORCES ABOUT A POINT IN THEIR PLANE.

1. The algebraical sum of the moments of any two forces about a point in their plane is equal to the moment of their resultant about the same point.

2. Extend the theorem to any number of forces in one plane acting on a rigid body.

3. If any number of coplanar forces act on a rigid body, and if the algebraical sum of their moments about each of two points in the plane of the forces be the same, show that the resultant of the forces acts along a line parallel to that joining the points.

4. If any number of coplanar forces act on a rigid body, and if the algebraical sum of the moments of these forces about each of three non-collinear points in their plane be zero, show that the forces are in equilibrium.

5. Two forces, P and Q , acting at an angle α at a point O , are represented by OA and OB , and the parallelogram $OACB$ is completed; if the direction of the resultant R make an angle θ with OA , show, by taking moments about A and B and eliminating θ from the two equations, that

$$R^2 = P^2 + Q^2 + 2PQ \cos \alpha.$$

6. Forces P , Q , R act along the altitudes AO , BO , CO of a triangle; if the forces be in equilibrium, show that

$$P : Q : R = a : b : c.$$

7. A system of forces X , Y , Z acting along the sides of the triangle ABC is equivalent to a system P , Q , R along the sides of the pedal triangle; prove that

$$2P = Y/\cos B + Z/\cos C.$$

8. Three given forces act along the altitudes of a triangle. Find the condition that the line of action of the resultant should pass through the centroid of the triangle, and show that, if the forces be equal, the triangle must be isosceles.

67.

THE ASYMPTOTES OF AN HYPERBOLA.

1. Find the equation giving the abscissae of the points of intersection of the line $y = mx + c$ and the hyperbola

$$a^2y^2 - b^2x^2 = -a^2b^2,$$

and deduce the equations of the asymptotes.

2. Find the equation of the tangent at (x', y') to the same hyperbola, and deduce the form of the equation when x' and y' are increased indefinitely.

3. Show that the distance of a point on the hyperbola from the corresponding asymptote continually decreases as the point moves away from the vertex, and hence determine the position of the curve with regard to both asymptotes.

4. Find the equation of an hyperbola referred to its asymptotes as axes.

5. Show that the portion of a tangent to an hyperbola intercepted between the asymptotes is bisected at the point of contact.

6. Show that the tangent to an hyperbola includes with the axes a triangle of constant area.

7. If one asymptote of the curve

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$$

pass through the origin, prove that

$$af^2 + bg^2 = 2fgh.$$

8. Find the equation of the hyperbola which has

$$x - y - 1 = 0 \quad \text{and} \quad x + y - 3 = 0$$

for asymptotes, and which passes through the origin.

68.

STATICAL PROOFS OF GEOMETRICAL THEOREMS.

1. Show, by considering the centre of gravity of equal masses placed at the angular points of a triangle, that the medians of a triangle are concurrent.

2. Show, by considering the centre of gravity of masses proportional to $\tan A$, $\tan B$, $\tan C$ at the angular points of a triangle ABC , that the altitudes of a triangle are concurrent.

3. Show, by considering the centre of gravity of masses proportional to the opposite sides placed at the angular points of a triangle, that the bisectors of the angles of a triangle are concurrent.

4. Show, by considering the centre of gravity of equal masses placed at the angular points of a quadrilateral, that the straight lines joining the mid-points of opposite sides and of the diagonals of a quadrilateral are concurrent and bisect one another.

5. Show, by considering the centre of gravity of equal masses placed at the angular points of a tetrahedron, that the straight lines which join the mid-points of opposite edges of a tetrahedron are concurrent and bisect one another.

6. Show, by considering the centre of gravity of equal masses placed at the angular points of a tetrahedron, that the lines joining each angular point to the centroid of the opposite face are concurrent.

7. $ABCD$ is a quadrilateral; AB , DC produced meet in E , and BC , AD in F ; the straight lines which bisect the angles A , C internally meet in X , those which bisect the angles B , D internally in Y , and those which bisect the angles E , F externally in Z ; by considering a system of four equal forces acting respectively along AB , AD , CB , CD , show that the points X , Y , Z are collinear.

8. $ABCD$ is a quadrilateral, AB , DC produced meet in E , BC , AD produced in F ; by considering the system of forces $\overline{AB}$, $\overline{AD}$, $\overline{CB}$, $\overline{CD}$, show that the resultant passes through the mid-point of each of the three diagonals AC , BD , EF , and hence that the mid-points of the diagonals are collinear.

69.

LOSS OF KINETIC ENERGY BY IMPACT.

1. If two spheres impinge directly, the total kinetic energy after impact is less than that before impact.

2. Prove the same theorem in the case of oblique impact.

3. A ball of mass M , moving with velocity u , impinges directly on a ball of mass m at rest; show that the amount of kinetic energy lost is

$$\frac{Mmu^2(1-e^2)}{2(M+m)}.$$

4. Two spheres, of masses 6 and 9 lb. and moving in the same direction with velocities of 20 and 12 ft. per sec., impinge directly; if the coefficient of restitution be $\frac{1}{2}$, find the loss of kinetic energy per cent. of the original amount.

5. If the mass of a particle which impinges directly upon another be $\frac{1}{100}$ of the mass of the latter, but if it have 100 times its velocity, show that the energy lost is only about

4 per cent. less than the original energy of the smaller particle, the coefficient of restitution being $\frac{1}{10}$.

6. If the momenta of two spheres impinging directly on one another be equal and opposite, show that the kinetic energy will be reduced by the impact in the ratio $e^2 : 1$.

7. A bullet of mass m lb. is fired with a horizontal velocity of u ft. per sec. into a mass of M lb. at rest suspended by a string, and is embedded in this mass. Find the energy lost at the impact.

Ex.—If $m = \frac{1}{2}$ oz., $M = 10$ lb., $u = 1800$ ft. per sec., find the energy lost in foot-pounds.

8. If two spheres impinge directly, show that if the one which has the greater mass has also the greater velocity, then the kinetic energies of the two spheres are more nearly equal after the impact than they were before.

70.

EQUATION OF A STRAIGHT LINE IN THE FORM

$$x = h + r \cos \theta, \quad y = k + r \sin \theta.$$

1. If (h, k) be a fixed point on a line inclined at an angle θ to the axis of x , and (x, y) a point on the line at distance r from the fixed point, show that

$$x = h + r \cos \theta, \quad y = k + r \sin \theta.$$

2. Lines are drawn through a point (h, k) to cut a circle

$$x^2 + y^2 + 2gx + 2fy + c = 0;$$

show that the rectangle contained by the segments is constant.

3. Find the locus of the mid-points of a series of parallel chords of a parabola.

4. If a point be taken on a series of parallel chords of a parabola so that the rectangle contained by the segments of

the chords is constant, show that the locus of the point is an equal parabola.

5. Find the equation of the chord of the parabola $y^2=8x$ which is bisected at the point $(2, -3)$.

6. Find the locus of the mid-points of chords of the hyperbola $xy=c^2$ which are parallel to the line $lx+my=0$.

7. Find the locus of the mid-points of chords of a circle which pass through a fixed point.

8. Prove that the locus of the mid-points of chords of the hyperbola $xy=c^2$ which are of constant length $2l$ is

$$(x^2+y^2)(xy-c^2)=l^2xy,$$

the axes being rectangular.

71.

THE FORMULAE FOR $\cos 7\theta$ AND $\sin 7\theta$ IN TERMS OF $\cos \theta$ AND $\sin \theta$.

1. Show that

$$\cos 7\theta = 64 \cos^7 \theta - 112 \cos^5 \theta + 56 \cos^3 \theta - 7 \cos \theta$$

2. Show that

$$\sin 7\theta = 7 \sin \theta - 56 \sin^3 \theta + 112 \sin^5 \theta - 64 \sin^7 \theta.$$

3. Putting 7θ in succession equal to $0, 2\pi, 4\pi, 6\pi, 8\pi, 10\pi, 12\pi$ in the equation of No. 1, show that $\cos \frac{2\pi}{7}, \cos \frac{4\pi}{7}, \cos \frac{6\pi}{7}, \cos \frac{8\pi}{7}, \cos \frac{10\pi}{7}$ and $\cos \frac{12\pi}{7}$ are the roots of the equation $64x^6 + 64x^5 - 48x^4 - 48x^3 + 8x^2 + 8x + 1 = 0$,

and hence that $\cos \frac{2\pi}{7}, \cos \frac{4\pi}{7}$ and $\cos \frac{8\pi}{7}$ are the roots of the equation $8x^3 + 4x^2 - 4x - 1 = 0$.

4. Hence, show that

$$\cos \frac{2\pi}{7} + \cos \frac{4\pi}{7} + \cos \frac{8\pi}{7} = -\frac{1}{2},$$

$$\cos \frac{4\pi}{7} \cos \frac{8\pi}{7} + \cos \frac{8\pi}{7} \cos \frac{2\pi}{7} + \cos \frac{2\pi}{7} \cos \frac{4\pi}{7} = -\frac{1}{2}$$

and
$$\cos \frac{2\pi}{7} \cos \frac{4\pi}{7} \cos \frac{8\pi}{7} = \frac{1}{8}.$$

5. Putting 7θ in succession equal to $\frac{\pi}{2}, \frac{5\pi}{2}, \frac{9\pi}{2}, \frac{13\pi}{2}, \frac{17\pi}{2}, \frac{21\pi}{2}, \frac{25\pi}{2}$ in the equation of No. 2, show that $\sin \frac{\pi}{14}, \sin \frac{5\pi}{14}$ and $\sin \frac{25\pi}{14}$ are the roots of the equation

$$8x^3 - 4x^2 - 4x + 1 = 0.$$

6. Hence, show that

$$\operatorname{cosec} \frac{\pi}{14} + \operatorname{cosec} \frac{5\pi}{14} + \operatorname{cosec} \frac{25\pi}{14} = 4.$$

7. Putting 7θ in succession equal to $0, 2\pi, 4\pi, 6\pi, 8\pi, 10\pi, 12\pi$ in the equation of No. 2, show that $\sin^2 \frac{2\pi}{7}, \sin^2 \frac{4\pi}{7}$ and $\sin^2 \frac{8\pi}{7}$ are the roots of the equation

$$64x^3 - 112x^2 + 56x - 7 = 0.$$

8. Hence, show that

$$\sin \frac{2\pi}{7} + \sin \frac{4\pi}{7} + \sin \frac{8\pi}{7} = \frac{1}{2}\sqrt{7},$$

$$\sin \frac{4\pi}{7} \sin \frac{8\pi}{7} + \sin \frac{8\pi}{7} \sin \frac{2\pi}{7} + \sin \frac{2\pi}{7} \sin \frac{4\pi}{7} = 0,$$

$$\sin \frac{2\pi}{7} \sin \frac{4\pi}{7} \sin \frac{8\pi}{7} = -\frac{1}{8}\sqrt{7}.$$

72.

COMPLEX QUANTITY. II.

1. Draw the Argand diagram $OABC$ of the complex expression

$$(1, \alpha) + (1, 2\alpha) + (1, 4\alpha),$$

where $\alpha = 2\pi/7$. Join OB , OC , produce BA to meet the initial line in D , and draw BM , CN perpendicular to the initial line. Show that

$$BO = BD \quad \text{and} \quad AD = OD.$$

2. In the same diagram, show that

$$OB = BA + OD,$$

and therefore that $2 \cos \frac{\alpha}{2} = 1 + 2 \cdot OM$

and $ON = OM - MN = -\frac{1}{2}.$

3. If $\alpha = 2\pi/7$, show that

$$\cos \alpha + \cos 2\alpha + \cos 4\alpha = -\frac{1}{2}.$$

4. In the figure of No. 1, show that

$$OC^2 = 1 + 4 \cos \frac{\alpha}{2} + 4 \cos^2 \frac{\alpha}{2} - 8 \cos^3 \frac{\alpha}{2}$$

and $2 \cos \frac{\alpha}{2} = 1 + \frac{1}{2} \sec \alpha,$

and hence that $OC^2 = 2.$

5. If $\alpha = 2\pi/7$, show that

$$\sin \alpha + \sin 2\alpha + \sin 4\alpha = \frac{1}{2}\sqrt{7}.$$

6. The roots of the quadratic equation $ax^2 + 2bx + c = 0$, where a, b, c are real and $b^2 < ac$, are represented on an Argand diagram by the points P, Q . Prove that P and Q are equidistant from the origin, and that PQ is perpendicular to the axis of real numbers.

7. Hence, show that the points P and Q may be found by a geometrical construction which does not require the solution of the equation.

8. Prove that, if a' , b' , c' be real and $b'^2 < a'c'$, the points representing the roots of the equation $a'x^2 + 2b'x + c' = 0$ lie on the circle through P , Q and the origin, if $bc' = b'c$.

73.

PTOLEMY'S THEOREM.

1. Prove that, in a cyclic quadrilateral, the sum of the rectangles contained by opposite pairs of sides is equal to the rectangle contained by the diagonals (Ptolemy's theorem).

2. (i) ABC is an equilateral triangle and P any point on the minor arc BC of the circumcircle; show that

$$PA = PB + PC.$$

(ii) A right-angled triangle ABC is inscribed in a circle, the right angle A is bisected by AD , which meets the circle again in D ; prove that

$$AB + AC = \sqrt{2} \cdot AD.$$

3. $ABCD$ is a quadrilateral inscribed in a circle; if AE be the chord parallel to BD , show that

$$AB \cdot BC + AD \cdot DC = BD \cdot CE.$$

4. If two circles cut orthogonally, the rectangle contained by their common chord and their line of centres is equal to twice the rectangle contained by their radii.

5. If $ABCD$ be a quadrilateral inscribed in a circle of which AC is a diameter, and if $\angle BAC = \alpha$, $\angle CAD = \beta$, show, by means of Ptolemy's theorem, that

$$\sin(\alpha + \beta) = \sin \alpha \cos \beta + \cos \alpha \sin \beta.$$

6. If $ABCD$ be a quadrilateral inscribed in a circle $ABEC$ of which AE is a diameter ; also if $\angle BAC = \alpha$, $\angle CAD = \beta$, show, by means of Ptolemy's theorem, that

$$\sin \alpha + \sin \beta = 2 \sin \frac{\alpha + \beta}{2} \cos \frac{\alpha - \beta}{2}.$$

7. Deduce the following from Ptolemy's theorem : An arc AB of a circle is divided equally in M and unequally in P ; prove that $PA \cdot PB + MP^2 = AM^2$,

and deduce that

$$\sin (\alpha + \beta) \sin (\alpha - \beta) = \sin^2 \alpha - \sin^2 \beta.$$

8. If a, b, c, d be the lengths of the sides AB, BC, CD, DA , and x, y the lengths of the diagonals BD, AC , of a cyclic quadrilateral, show that

$$x : y = \sin A : \sin B = ad + bc : ab + cd,$$

and hence, by Ptolemy's theorem, show that

$$x = \sqrt{\left\{ \frac{(ac + bd)(ad + bc)}{ab + cd} \right\}}, \quad y = \sqrt{\left\{ \frac{(ac + bd)(ab + cd)}{ad + bc} \right\}}.$$

74.

THE EVALUATION OF π .

1. Prove Huyghens' approximation for the length of a circular arc in the form $\frac{1}{3}(8B - A)$, where A is the chord of the arc and B that of half the arc.

2. Find the length of the arc of a circle of radius a subtending an angle of 20° at the centre ; and hence deduce the value of π to 4 places of decimals.

3. Prove Euler's product theorem

$$\frac{\sin \theta}{\theta} = \cos \frac{\theta}{2} \cos \frac{\theta}{2^2} \cos \frac{\theta}{2^3} \cdots,$$

and deduce Vieta's formula for π , namely,

$$\frac{2}{\pi} = \frac{\sqrt{2}}{2} \cdot \frac{\sqrt{2+\sqrt{2}}}{2} \cdot \frac{\sqrt{2+\sqrt{2+\sqrt{2}}}}{2} \dots$$

4. Find the value of π to 3 places of decimals from Vieta's formula.

5. AB is a diameter of a circle of radius unity and centre O ; BC , the tangent at B , is equal to three times the radius; AD , the tangent at A , is such that $\angle AOD = \pi/6$; show that the line CD is approximately equal to half the circumference of the circle.

6. AOB is the diameter of a given circle; from OA is cut off $OC = \frac{3}{5}OA$; from OB produced, $OD = \frac{3}{2}OA$; OB is bisected in E ; semicircles CFE , AGD are described, and the perpendicular through O to AB cuts them in F , G respectively; show that FG agrees to 4 places of decimals with the side of a square, the area of which is equal to that of the circle.

7. Show that

$$(i) \quad \frac{\pi}{4} = 4 \tan^{-1} \frac{1}{5} - \tan^{-1} \frac{1}{239} \quad (\text{Machin}),$$

$$(ii) \quad \frac{\pi}{4} = 5 \tan^{-1} \frac{1}{7} + 2 \tan^{-1} \frac{3}{79} \quad (\text{Euler}).$$

8. Assuming Gregory's series, namely,

$$\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \dots,$$

where $|x| < 1$, find, by Machin's formula, the value of π to 10 places of decimals.

75.

EULER'S PRODUCT THEOREM.

1. Prove that

$$\sin \theta = 2^{n-1} \cos \frac{\theta}{2} \cos \frac{\theta}{2^2} \cos \frac{\theta}{2^3} \dots \cos \frac{\theta}{2^n} \sin \frac{\theta}{2^n}.$$

2. Prove that

$$\lim_{n \rightarrow \infty} \left(\cos \frac{\theta}{2} \cos \frac{\theta}{2^2} \dots \cos \frac{\theta}{2^n} \right) = \frac{\sin \theta}{\theta} \quad (\text{Euler's theorem}).$$

3. From the preceding equation, deduce Vieta's expression for π , namely,

$$\frac{2}{\pi} = \frac{\sqrt{2}}{2} \cdot \frac{\sqrt{2+\sqrt{2}}}{2} \cdot \frac{\sqrt{2+\sqrt{2+\sqrt{2}}}}{2} \dots$$

4. By means of Euler's theorem, show that, if $\theta < \pi$, $\sin \theta / \theta$ decreases as θ increases.

5. If $u_1, u_2, u_3, \dots$ be positive quantities less than unity, show that

$$(i) (1 - u_1)(1 - u_2) > 1 - (u_1 + u_2)$$

$$\text{and} \quad (ii) (1 - u_1)(1 - u_2)(1 - u_3) > 1 - (u_1 + u_2 + u_3).$$

Hence, if the infinite series $u_1 + u_2 + u_3 + \dots$ be convergent, show that the infinite product

$$(1 - u_1)(1 - u_2)(1 - u_3) \dots$$

is greater than

$$1 - (u_1 + u_2 + u_3 \dots).$$

6. Hence, making use of the fact that $\cos \theta > 1 - \frac{1}{2}\theta^2$, θ being an acute angle, show that $\sin \theta > \theta - \frac{1}{6}\theta^3$.

7. Prove that

$$\lim_{n \rightarrow \infty} \left(1 - \tan^2 \frac{\theta}{2} \right) \left(1 - \tan^2 \frac{\theta}{2^2} \right) \left(1 - \tan^2 \frac{\theta}{2^3} \right) \dots \left(1 - \tan^2 \frac{\theta}{2^n} \right) = \theta / \tan \theta.$$

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8. If a, b be positive quantities, and if

$$a_1 = \frac{1}{2}(a+b), \quad b_1 = (a_1 b)^{\frac{1}{2}}, \quad a_2 = \frac{1}{2}(a_1 + b_1), \quad b_2 = (a_2 b_1)^{\frac{1}{2}},$$

and so on, prove that

$$\lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} b_n = (b^2 - a^2)^{\frac{1}{2}} / \cos^{-1} a/b.$$

76.

GRAPHS AND THE SOLUTION OF TRIGONOMETRICAL EQUATIONS.

1. Show, by drawing the graphs of the curves, (i) that the equation $\sin x = c$ has two roots between 0 and π if $c < 1$, and no roots between those values if c be negative or > 1 ; (ii) that the equation $\cos \theta = c$ has only one root between 0 and π if c lie between -1 and $+1$, and no root between those values if $c < -1$ or > 1 .

2. Show that the equation $\cos \theta = \theta$ has only one real root, and that it lies between 0 and $\pi/4$.

3. Show that the equation $\tan \theta = m\theta$ has an infinite number of real roots for all values of m . Also, if θ lie between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$, show that the equation has three real and different roots, three real and equal roots, or only one real root, according as m is greater than, equal to, or less than 1; while the equation $\cot \theta = m\theta$ has two real solutions in every case.

4. Show graphically that the equation $\sin 2\theta = \sin \theta$ has three roots between 0 and 2π .

5. Draw the curve $y = \cot x$, and hence calculate roughly the roots of the equation $x \cot x = 1 - 2\pi x$.

6. Show that the equation $\sin \theta + \cos \theta = c$ has two real roots between 0 and 2π if c lie between $-\sqrt{2}$ and $+1$, and three real roots if c lie between 1 and $\sqrt{2}$.

7. Show graphically that

(i) the equation $\sin \theta + \cos \theta = \sqrt{2} \cdot \sin 2\theta$ has three roots between 0 and 2π ;
 (ii) the equation $\sec \theta + \operatorname{cosec} \theta = c$ has two roots between 0 and 2π if $c^2 < 8$, and four roots if $c^2 > 8$.

8. Show that the graph of the expression

$$\cos px + \cos qx$$

lies between those of $2 \cos \frac{p-q}{2}x$ and $-2 \cos \frac{p-q}{2}x$, touching each in turn. What general conclusion can be drawn as to the amplitudes of the variations of $\cos px + \cos qx$ when $p-q$ is small compared with $p+q$?

77.

BROCARD POINTS.

1. Describe a circle so as to touch a given straight line AB at A and to pass through a point C outside AB .

2. If ABC be a triangle, and if two circles be drawn, one to touch AB at A and to pass through C , the other to touch CA at C and to pass through B , and if the circles intersect in O , show that $\angle BAO = \angle ACO = \angle CBO$.

3. Hence, show that, if ABC be a triangle, the three circles which touch AB at A , BC at B and CA at C , and pass respectively through the points C , A , B , pass through one point.

4. If this point O be called a *Brocard point*, show that every triangle has two Brocard points.

5. If each of the equal angles BAO , ACO , CBO be denoted by ω , show that

$$\sin^3 \omega = \sin(A - \omega) \sin(B - \omega) \sin(C - \omega).$$

6. Dividing both sides of the preceding equation by $\sin^3 \omega$, show that the equation reduces to the following cubic equation in $\cot \omega$:

$$\begin{aligned} \cot^3 \omega \cdot \sin A \sin B \sin C - \cot^2 \omega (\Sigma \cos A \sin B \sin C) \\ \cot \omega \cdot \sin A \sin B \sin C - \Sigma \cos A \sin B \sin C = 0. \end{aligned}$$

7. Hence, show that

$$\cot \omega = \cot A + \cot B + \cot C.$$

8. Hence, show that

$$\operatorname{cosec}^2 \omega = \operatorname{cosec}^2 A + \operatorname{cosec}^2 B + \operatorname{cosec}^2 C.$$

78.

THE INEQUALITY THEOREM $\sin \theta > \theta - \frac{1}{6}\theta^3$.

1. Show that

$$\sin \theta = 3^n \sin \frac{\theta}{3^n} - 4 \left(\sin^3 \frac{\theta}{3} + 3 \sin^3 \frac{\theta}{3^2} + 3^2 \sin^3 \frac{\theta}{3^3} + \dots + 3^{n-1} \sin^3 \frac{\theta}{3^n} \right).$$

2. Hence, show that, if $\theta < \frac{\pi}{2}$,

$$\sin \theta > \theta - 4 \left(\frac{\theta^3}{3^3} + \frac{\theta^3}{3^5} + \frac{\theta^3}{3^7} + \dots \text{to inf.} \right),$$

and therefore that $\sin \theta > \theta - \frac{1}{6}\theta^3$.

3. If AB be an arc of a circle subtending an angle $\theta \left(< \frac{\pi}{2} \right)$ at the centre of the circle, and if $C, D, E, F, \dots$ be the mid-points of the arcs $AB, AC, AD, AE, \dots$, prove that

$$\text{sector } AOB - \triangle AOB = \triangle ACB + 2\triangle ADC + 2^2\triangle AED + \dots$$

4. If a be the radius of the circle and if $\angle AOB$ contain θ radians, show that the area of the triangle ACB is

$$2a^2 \sin \frac{\theta}{2} \sin^2 \frac{\theta}{2},$$

and deduce that of the triangle ADC .

5. Hence, show that $\triangle ACB < \frac{a^2 \theta^3}{24}$, $ADC < \frac{a^2 \theta^3}{27}$, and therefore that

$$\frac{1}{2}a^2(\theta - \sin \theta) < \frac{a^2 \theta^3}{24} \left(1 + \frac{1}{2^2} + \frac{1}{2^4} + \dots\right),$$

and therefore that $\sin \theta > \theta - \frac{1}{6}\theta^3$.

6. Prove that

$$\sin \theta = 2^n \sin \frac{\theta}{2^n} \cos \frac{\theta}{2} \cos \frac{\theta}{2^2} \dots \cos \frac{\theta}{2^n},$$

and hence, by increasing n indefinitely, deduce Euler's product theorem that

$$\sin \theta / \theta = \cos \frac{\theta}{2} \cos \frac{\theta}{2^2} \cos \frac{\theta}{2^3} \dots$$

7. If u_1 and u_2 be proper fractions, prove that

$$(1 - u_1)(1 - u_2) > 1 - (u_1 + u_2)$$

and $(1 - u_1)(1 - u_2)(1 - u_3) > 1 - (u_1 + u_2 + u_3)$,

and extend the theorem to the product of an infinite series of factors.

8. Assuming that $\cos \theta > 1 - \frac{1}{2}\theta^2$, where $\theta < \frac{\pi}{2}$, show, by means of the preceding theorem, that $\sin \theta > \theta - \frac{1}{6}\theta^3$.

79.

THE SIMSON LINE.

1. The feet of the perpendiculars drawn to the sides of a triangle from any point on its circumcircle are collinear. (The line passing through the feet of the perpendiculars is

called the *Simson line* of the point with respect to the given triangle.)

2. If P be a point on the circumcircle of a triangle, and if PD , PE , PF , the perpendiculars to the sides, meet the circumcircle in X , Y , Z , show that AX , BY , CZ are parallel to the Simson line.

3. The angle between the Simson lines of any two points on a circle with respect to any inscribed triangle is equal to half the angle subtended by the arc between the points at the centre of the circle.

Hence, show that (i) the Simson lines of the ends of a diameter of a circle with respect to any inscribed triangle are perpendicular to one another, and (ii) the Simson lines of three points P , Q , R on the circle form by their intersections a triangle similar to the triangle PQR .

4. If the Simson line of a point on the circumcircle of the triangle ABC be equally inclined to two sides, prove that the point bisects the arc intercepted by the third side.

5. If, from a point P on the circumcircle of a triangle ABC , PD be drawn perpendicular to BC and be produced to meet the circumcircle in X , and if the Simson line of the point P meet the altitude from A produced in K , then $DX=AK$.

6. Also, if O be the orthocentre of the triangle ABC , show that $PD=KO$. Hence, show that OP is bisected by the Simson line.

7. Show that the locus of the mid-point of a line drawn from the orthocentre of a triangle to the circumcircle is the nine-points circle of the triangle.

8. Hence, show that the Simson lines of the ends of the diameter of a circle with respect to any inscribed triangle

intersect in a point the locus of which is the nine-points circle of the triangle.

80.

INTERSECTING SPHERES.

1. Prove that the common section of two intersecting spheres is a circle.

2. Two spheres of radii a and b intersect, the distance between their centres being c ; find the area of their common section.

3. A line parallel to the line of centres of two intersecting spheres passes round the common section of the spheres; show that this line traces out on each sphere a circle the plane of which is perpendicular to the line of centres.

4. Prove that two concentric spheres intercept a constant area on any sphere which passes through their common centre.

5. Prove that all spheres cutting two given non-intersecting spheres at right angles have their centres on a fixed plane and pass through two fixed points.

6. A line parallel to the line of centres of two spheres which cut orthogonally passes round the curve of intersection of the spheres; show that the volume of the right circular cylinder inscribed within the complete figure is

$$2\pi a^2 b^2 / \sqrt{(a^2 + b^2)},$$

where a, b are the radii of the spheres.

7. There are two spheres of radii a, b , with their centres at a distance $c (< a + b)$ apart. Find the area of that part of the surface of the sphere of radius a which lies inside the other sphere.

8. A closed surface is formed by two spheres of radii a , b , which cut one another at right angles. Show that the external surface area is $2\pi(a^3 + b^3 + c^3)/c$, where c is the distance between the centres.

81.

SPHERICAL SEGMENTS.

1. Find the volume of a spherical segment of one base, the height of the segment being a and the radius of the sphere r . Deduce the volume of a spherical segment with two bases.

2. Each of two equal spheres passes through the centre of the other; show that the volume common to the two spheres is $\frac{5}{12}$ of the volume of either.

3. A sphere floats in water, so that the height of the cap above water is one-tenth of the diameter; find the density of the sphere.

4. If the height a of a spherical segment be very small compared with the radius r of the sphere, show that its volume is approximately $\pi a^2 r$.

Hence, find approximately the volume of a lens, of which the diameter is one inch and the radii of its surfaces (supposed spherical) respectively 25 ins. and 20 ins.

5. If a slice be cut off from a sphere by two parallel planes, prove that its volume is

$$\frac{1}{6}\pi t^3 + \frac{1}{2}t(A + B),$$

where t is the thickness of the slice, and A , B the areas of its plane faces.

6. A cylindrical hole is bored through a sphere with its axis coinciding with a diameter of the sphere; prove that the

volume of the remainder is equal to that of a sphere, the diameter of which is equal to the length of the hole.

7. A hemispherical bowl full of water is tilted slowly through an angle of 30° ; find what percentage of the water overflows.

8. A segment of a circle revolves about any diameter not intersecting it; prove that the volume generated by it is one-sixth of that of a circular cylinder, the radius of which is equal to the chord and the altitude of which is equal to the projection of the chord on the diameter.

82.

LINES OF QUICKEST DESCENT.

1. The time of sliding from rest down any chord of a vertical circle drawn from the highest point or to the lowest point is constant.

Hence, show that the line of quickest descent from a point to a curve in the same vertical plane is such that a vertical circle with the given point for its highest point touches the curve at the point at which the line of quickest descent meets it.

2. Find the line of quickest descent from a given point to a given straight line in the same vertical plane as the point.

3. Find the line of quickest descent from a given point to a given circle in the same vertical plane as the point.

4. A right-angled triangle is placed so that one of the sides containing the right angle is vertical. Determine the position of the point in the hypotenuse from which the time of the particle's descent to the right angle may be the least possible.

5. A circle has a vertical diameter AB , and two particles slide down two chords AP , PB respectively, starting simultaneously from A , P ; prove that the least distance between them during the motion is equal to the distance of P from AB .

6. Show that the shortest time of descent down a smooth straight tube from a circle to a point in the same vertical plane is

$$2g^{-\frac{1}{2}}\{r \tan \alpha \cot (\beta - \alpha)\}^{\frac{1}{2}},$$

where r is the radius of the circle, and the lines drawn from the given point to the highest and lowest points of the circle make respectively with the vertical the angles α and β .

7. Show that the chords of quickest and slowest descent from the highest point of an ellipse in a vertical plane are at right angles to each other and parallel to the axes of the curve.

8. Two parabolas are placed in the same vertical plane with their foci coincident, axes vertical and vertices downwards. Prove that the chord of quickest descent from the outer to the inner parabola passes through the focus and makes an angle $\pi/3$ with the axis.

83.

CENTRE OF GRAVITY OF A CIRCULAR ARC AND SECTOR.

1. Find the centre of gravity of a circular arc, and deduce that of a semicircle.

2. Given the centre of gravity of a circular arc, deduce that of a sector of a circle.

3. Deduce the position of the centre of gravity of a segment of a circle.

4. Deduce the position of the centre of gravity of a semicircle from each of the results of Nos. 2 and 3.

5. A semicircle is described with radius a , and with one extremity of this semicircle as centre and radius $2a$ another semicircle is described so as to form with the former a continuous curve. Show that the centre of gravity of a wire in the form of this curve is distant $\frac{3a}{2\pi}\sqrt{(36+\pi^2)}$ from the centre of gravity of the smaller semicircle.

6. Two tangents are drawn to a circle of radius a subtending an angle 2α at the centre. Prove that the centre of gravity of the figure bounded by the tangents and the smaller arc between them is at a distance from the centre equal to

$$\frac{a \tan^2 \alpha \sin \alpha}{3 \tan \alpha - \alpha}.$$

7. ACB is a semicircle of radius a , C being the middle point of the arc, and PM , QN are drawn from points P , Q on the arc perpendicular to the diameter AB . Show that, if PC QC each subtend an angle α at the centre of the circle, the centre of gravity of the area bounded by the arc PCQ and the straight lines PM , MN , NQ will be at a distance

$$\frac{a \sin \alpha (2 + \cos^2 \alpha)}{3(a + \sin \alpha \cos \alpha)}$$

from the centre of the circle.

8. Parallel tangents are drawn to the inner of two concentric circles of radii a and $a\sqrt{2}$. Show that the centre of gravity of either of the areas bounded by the parallel tangents and the arcs of the circles between them is on the inner circle.

84.

THE SIGN OF THE DIFFERENTIAL COEFFICIENT.

1. If y be a function of x , define the differential coefficient of y with respect to x , and illustrate the definition by a diagram. Explain what light the sign of the differential coefficient throws on the increase or decrease of y as x increases.

2. Show, by considering the sign of the differential coefficients of the following expressions with regard to x , that
 (i) $x^2 - 6x + 8$ decreases until $x=3$ and then increases, and
 (ii) $3 + 8x - x^2$ increases as x increases until $x=4$ and then decreases. Illustrate both facts by diagrams.

3. Draw an arc at every point of which x , y , $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ are all positive, and another arc at every point of which the same quantities are all negative; and give reasons for so drawing them.

4. From the corners of a square piece of cardboard, of side 12 ins., squares are cut away and the projecting portions folded up so as to form a box without a lid; find the maximum volume of the box.

5. Show that the expression

$$x^3 - 6x^2 + 21x + 1$$

is positive for all positive values of x .

6. If x be greater than 0 and less than 1, show, by means of the differential coefficients of the expressions

$$x - \log(1+x) \quad \text{and} \quad x - \frac{x^2}{2} - \log(1+x),$$

that $\log(1+x) < x$ and $> x - \frac{x^2}{2}$.

7. Show that a curve is concave or convex towards the positive direction of the y -axis according as $\frac{d^2y}{dx}$ is positive or negative. Hence, show that, at a point of inflexion, $\frac{d^2y}{dx^2}$ changes sign.

8. Show that the curve

$$y = x^3 - 12x^2 + 6x + 104$$

has a point of inflexion at the point (4, 0).

85.

MAXIMA AND MINIMA TREATED ALGEBRAICALLY.

1. Find the maximum value of $7 + 10x - x^2$ and the minimum value of $x^2 + 11x + 40$.

2. If $5x + 4y = 200$, find the maximum value of xy .

3. Find the maximum and minimum values of

$$\frac{x^2 - x + 1}{x^2 + x + 1}.$$

4. Find the maximum value of

$$\frac{2u^2 \cos \alpha \sin (\alpha - \beta)}{g \cos^2 \beta},$$

where u and β are constant.

5. Find the maximum and minimum values of

$$a \cos \theta + b \sin \theta.$$

6. Find the maximum and minimum values of

$$a \cos^2 \theta + 2h \cos \theta \sin \theta + c \sin^2 \theta.$$

7. If ABC be a triangle, show that the minimum value of

$$\tan^2 \frac{A}{2} + \tan^2 \frac{B}{2} + \tan^2 \frac{C}{2} \text{ is } 1.$$

8. If ABC be a triangle and P a point in its plane, show that

$$AP^2 \tan A + BP^2 \tan B + CP^2 \tan C$$

is a minimum when P coincides with the orthocentre of the triangle.

86.

MAXIMA AND MINIMA TREATED GEOMETRICALLY.

1. Show that, on either side of a turning value of a function, there are two values of the function that are equal to one another.

2. Show that the greatest and least straight lines which can be drawn to an oval curve from any point are normal to the curve.

3. Of all chords of an ellipse which pass through a given point within it, that which divides the ellipse into maximum and minimum segments is bisected at the given point.

4. Show that the least triangle which can be described about a given circle is equilateral.

5. If a triangle of maximum area be inscribed in any oval curve, show that the tangent at each angular point is parallel to the opposite side.

6. If the normal at any point P of a curve cut the curve again in the point Q , prove that, if the length PQ be a maximum or minimum, either Q is the centre of curvature at P , or PQ is also a normal at Q .

7. If $\phi(x)$ have a turning value for any value of x , then $f\{\phi(x)\}$ has also a turning value for the same value of x .

8. Apply the method of No. 7 to determine the value of θ for which $\sec^3 \theta \operatorname{cosec} \theta$ is a minimum.

87.

DIAGONALS OF A CYCLIC QUADRILATERAL.

1. If $ABCD$ be a cyclic quadrilateral, a, b, c, d the lengths of the sides AB, BC, CD, DA , and x, y of the diagonals BD, AC , show, by equating the values of the area of the quadrilateral in terms of $\sin A$ and $\sin B$, that

$$x : y = \sin A : \sin B = ab + cd : ad + bc.$$

2. Hence, by means of Ptolemy's theorem ($xy = ac + bd$), show that

$$x = \sqrt{\left\{ \frac{(ab + cd)(ac + bd)}{ad + bc} \right\}}, \quad y = \sqrt{\left\{ \frac{(ac + bd)(ad + bc)}{ab + cd} \right\}}.$$

3. If the sides AB, DC of a cyclic quadrilateral $ABCD$ meet in E , and the sides BC, AD in F , prove that the circles BCE, CDF intersect in a point which lies on the third diagonal EF .

4. Hence, show that

$$EF^2 = EA \cdot EB + FA \cdot FD.$$

5. Show that

$$EA = \frac{d \sin B}{\sin(A + D)} \quad \text{and} \quad EB = \frac{b \sin A}{\sin(A + D)},$$

and therefore that $EA : EB = dy : bx$.

6. Hence, show that

$$EA = \frac{ady}{dy - bx}, \quad EB = \frac{abx}{dy - bx},$$

and therefore that

$$EA \cdot EB = \frac{a^2 bd \cdot xy}{(dy - bx)^2} \quad \text{and} \quad FA \cdot FD = \frac{b^2 ac \cdot xy}{(ay - cx)^2}.$$

7. From the ratios

$$x : y = ab + cd : ad + bc,$$

show that $\frac{x}{ab + cd} = \frac{y}{ad + bc} = \frac{dy - bx}{a(d^2 - b^2)} = \frac{ay - cx}{d(a^2 - c^2)}$

8. If z be the length of the third diagonal EF , show that

$$z^2 = (ab + cd)(ad + bc) \left[\frac{bd}{(b^2 - d^2)^2} + \frac{ac}{(a^2 - c^2)^2} \right]. \quad \cdot \quad \cdot$$

88.

THE SUM OF EQUIDISTANT TERMS OF A SERIES.

1. Prove that the product of the complex quantities $(1, \alpha)$ and $(1, \beta)$ is $(1, \alpha + \beta)$, and extend the theorem to the product of n complex quantities each equal to $(1, \alpha)$.

2. Find the n n th roots of unity, and show that, if ω denote any one of the complex n th roots of unity, the series of roots may be denoted by $\omega^0, \omega^1, \omega^2, \dots, \omega^{n-1}$.

3. If ω denote one of the complex n th roots of unity, show that

$$1 + \omega + \omega^2 + \dots + \omega^{n-1} = 0.$$

4. The sum of the m th powers of the n n th roots of unity is n or zero, according as m is or is not a multiple of n .

5. If c_r be the coefficient of x^r in the expansion of $(1+x)^n$, show that

$$c_0 + c_2 + c_4 + \dots = 2^{n-1}.$$

Also show that $1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \dots = \frac{1}{2}(e^x + e^{-x})$.

6. Find the sum of every third term of the series

$$1 + 2x + 3x^2 + 4x^3 + \dots,$$

beginning with (i) the first term, (ii) the second term, and (iii) the third term; and show that the sum of the three series is $(1-x)^{-2}$.

7. If $|x| < 1$, find the sum of the infinite series

$$1 \cdot 2 + 5 \cdot 6x^4 + 9 \cdot 10x^8 + \dots$$

8. If m be a positive integer and s the sum of the first, fourth, seventh, ... coefficients of the binomial expansion $(a+b)^n$, show that

$$3s = 2^m + (-1)^m \omega^m (1 + \omega^m),$$

where ω is a complex cube root of unity. Hence, show that s has one of the values

$$\frac{1}{3}(2^m \pm 2), \quad \frac{1}{3}(2^m + 1), \quad \frac{1}{3}(2^m - 1),$$

and distinguish the cases.

89.

INCYCLIC AND CIRCUMCYCLIC QUADRILATERALS.

1. If a quadrilateral be incyclic, prove that each pair of opposite angles is equal to two right angles; if it be circumcyclic, that the sum of one pair of opposite sides is equal to the sum of the other pair.

2. The area of an incyclic and circumcyclic quadrilateral is

$$\sqrt{(abcd)}, \quad \text{and} \quad \tan^2 \frac{A}{2} = \frac{bc}{ad}.$$

3. A quadrilateral can have one circle inscribed in it, and another described about it; show that the straight lines joining the opposite points of contact of the incircle are at right angles.

4. The diagonals of a quadrilateral which is inscribed in a circle subtend acute angles θ and ϕ at the circumference. If a circle can be inscribed in this quadrilateral, prove that the acute angle between the diagonals is

$$\tan^{-1} \left(\frac{\sin \theta + \sin \phi}{\cos \theta \cos \phi} \right).$$

5. If a quadrilateral can be inscribed in one circle and described about another, the radius of the latter circle is

$$\sqrt{abcd}/(a+c).$$

6. Show also that the radius of the larger circle is

$$\frac{1}{4} \sqrt{\left\{ \frac{(ab+cd)(ac+bd)(ad+bc)}{abcd} \right\}}.$$

7. If c, c' be the diagonals of a quadrilateral which is incyclic and circumcyclic, D, d the diameters of the circumcircle and incircle respectively, prove that

$$\frac{cc'}{d^2} - \frac{D^2}{cc'} = 1.$$

8. A quadrilateral is such that it can be inscribed in one circle and described about another. If R, r be the radii of these circles, and δ the distance between their centres, prove that

$$\frac{1}{(R+\delta)^2} + \frac{1}{(R-\delta)^2} = \frac{1}{r^2}.$$

90.

FEUERBACH'S THEOREM.

1. If S be the circumcentre, I the incentre, and O the orthocentre of the triangle ABC , show that $\angle SAO = \angle C \sim \angle B$, and that AI bisects the angle SAO .

2. Show that

$$SO^2 = R^2(1 - 8 \cos A \cos B \cos C).$$

Hence, show that, if A, B, C be the angles of a triangle, the greatest value of $\cos C \cos B \cos C$ is $\frac{1}{8}$.

3. Show that

$$SI^2 = R^2 - 2Rr.$$

Hence, show that, if A, B, C be the angles of a triangle, the greatest value of $\sin \frac{A}{2} \sin \frac{B}{2} \sin \frac{C}{2}$ is $\frac{1}{8}$.

4. If I_1 be the excentre opposite the angle A , show that

$$SI_1^2 = R^2 + 2Rr_1.$$

5. Show that $SI^2 + SI_1^2 + SI_2^2 + SI_3^2 = 12 \cdot R^2$.

6. Show that

$$(i) \quad OI^2 = 2r^2 - 4R^2 \cos A \cos B \cos C,$$

$$(ii) \quad OI_1^2 = 2r_1^2 - 4R \cos A \cos B \cos C.$$

7. If V be the nine-points centre of the triangle, show that

$$IV = \frac{1}{2}R - r \quad \text{and} \quad I_1V = \frac{1}{2}R + r_1.$$

8. If the distance between the centres of two circles be equal to the sum or difference of their radii, show that the circles touch one another externally or internally.

Hence, show that the nine-points circle of a triangle touches the incircle and excircles.

91.

RELATIONS BETWEEN THE CIRCUMCIRCLE AND EXCIRCLES OF A TRIANGLE.

1. Prove that $r_1 = 4R \sin \frac{A}{2} \cos \frac{B}{2} \cos \frac{C}{2}$, etc.

2. Prove that $r_1 + r_2 + r_3 = 4R + r$.

Hence, show that the radius of an excircle of an equilateral triangle is $3R/2$.

3. Show that all the excentres of a triangle lie outside the circumcircle.

4. Show that the circumcircle of a triangle cuts every excircle.

5. If P be one point of intersection of the circumcircle and the excircle opposite A , and if I_1P produced cut the circumcircle again in Q , show that I_1Q is equal to the diameter of the circumcircle.

6. Show that, in a triangle ABC , the length of the common chord of the circumcircle and the excircle opposite A is

$$\left\{ \frac{r_1^3(4R - r_1)}{R(R + 2r_1)} \right\}^{\frac{1}{2}}.$$

7. Show that the cosine of the angle at which the circumcircle of a triangle ABC is cut by the excircle opposite A is

$$\frac{1}{2}(1 + \cos A - \cos B - \cos C).$$

8. Hence, show that the circumcircle and the excircle opposite A cut orthogonally if $r_1 = 2R$.

92.

THE COMMON TANGENTS OF THE EXCIRCLES OF A TRIANGLE.

1. The lengths of the external and internal common tangents to the excircles opposite B and C of the triangle ABC are respectively $b + c$ and a .

2. The angle between the internal common tangents to the excircles opposite B and C of the triangle ABC is equal to A , the angles between an internal common tangent and either external common tangent are equal to B and C respectively, and the angle between the external common tangents is equal to $C \sim B$.

3. The external common tangents to the three pairs of excircles of a triangle ABC form a triangle XYZ ; prove that the angles of this triangle are the supplements of $2A, 2B, 2C$.

4. Prove the preceding theorem by showing that the sides of the triangle XYZ are parallel to those of the pedal triangle of the given triangle.

5. Hence, show that

$$\frac{YZ}{a \cos A} = \frac{ZX}{b \cos B} = \frac{XY}{c \cos C}.$$

6. Prove that the portions of the sides produced of the triangle ABC intercepted by the sides of the triangle XYZ are each equal to the perimeter of the triangle ABC .

7. Prove that the areas of the triangles cut off from the triangle XYZ by the sides produced of the given triangle are respectively $s^2 \tan A$, $s^2 \tan B$, $s^2 \tan C$.

8. Hence, show that the area of the triangle XYZ is

$$s^2 \tan A \tan B \tan C + 2 \triangle ABC.$$

93.

RELATION CONNECTING THE SIX DISTANCES BETWEEN FOUR POINTS.

1. If $\theta + \phi + \psi = 2\pi$, prove that

$$\cos^2 \theta + \cos^2 \phi + \cos^2 \psi - 2 \cos \theta \cos \phi \cos \psi = 1.$$

2. ABC is a triangle, P any point either within or without the triangle, α, β, γ the distances of P from the points A, B, C respectively, θ, ϕ, ψ the angles between each pair of the lines PA, PB, PC , so that

$$\theta + \phi + \psi = 2\pi ;$$

deduce from the formula

$$\cos^2 \theta + \cos^2 \phi + \cos^2 \psi - 2 \cos \theta \cos \phi \cos \psi = 1,$$

$$\begin{aligned} a^2 \alpha^2 (a^2 + \alpha^2 - b^2 - c^2 - \beta^2 - \gamma^2) + b^2 \beta^2 (b^2 + \beta^2 - c^2 - a^2 - \gamma^2 - \alpha^2) \\ + c^2 \gamma^2 (c^2 + \gamma^2 - a^2 - b^2 - \alpha^2 - \beta^2) + a^2 \beta^2 \gamma^2 + b^2 \gamma^2 \alpha^2 + c^2 \alpha^2 \beta^2 \\ + a^2 b^2 c^2 = 0. \end{aligned}$$

3. Verify the preceding relation when the four points are the angular points of a square.

4. Deduce from the formula in No. 2 the radius of the circumcircle of an equilateral triangle.

5. If x, y, z be the distances of any point in the plane of an equilateral triangle of side a from the angular points, show that

$$y^2z^2 + z^2x^2 + x^2y^2 - x^4 - y^4 - z^4 + a^2(x^2 + y^2 + z^2) - a^4 = 0.$$

6. In the plane of the triangle ABC and on the same side of BC as A , a point P is taken so that the angle BPC is $\frac{\pi}{2} + A$; if PA, PB, PC be denoted by α, β, γ respectively, show that

$$a^2\alpha^2 = b^2\beta^2 + c^2\gamma^2.$$

7. Three circles touching each other externally are all touched by a fourth circle including them all. If a, b, c be the radii of the three interior circles, and α, β, γ the distances of their centres from that of the external circle, prove that

$$2\left(\frac{\beta\gamma}{bc} + \frac{\gamma\alpha}{ca} + \frac{\alpha\beta}{ab}\right) = 4 + \frac{\alpha^2}{a^2} + \frac{\beta^2}{b^2} + \frac{\gamma^2}{c^2}.$$

8. Two points A, B are taken within a circle of radius ρ and centre C . Prove that the diameters of the circles which can be drawn through A and B to touch the given circle are the roots of the equation

$$x^2(\rho^2c^2 - a^2b^2\sin^2 C) - 2x\rho c^2(\rho^2 - ab\cos C) + c^2(\rho^4 - 2\rho^2ab\cos C + a^2b^2) = 0,$$

where the symbols refer to the parts of the triangle ABC .

94.

VALUES OF $\sum_{r=0}^{n-1} \sec\left(\theta + r \cdot \frac{2\pi}{n}\right)$ AND $\sum_{r=0}^{n-1} \sec^2\left(\theta + r \cdot \frac{2\pi}{n}\right)$.

1. If $u_n = \sin n\theta / \sin \theta$, $v = 2 \cos \theta$, prove that

$$u_{n+1} = u_n v - u_{n-1}.$$

Hence, find the values of $u_1, u_2, u_3, \dots u_8$ in terms of v .

2. Show that the coefficient of the $(r+1)$ th term of u_n is $(n-r-1)_r$, and hence that

$$u_n = v^{n-1} - (n-2)_1 v^{n-3} + (n-3)_2 v^{n-5} - \dots \\ + (-1)^r (n-r-1)_r v^{n-2r-1} + \dots$$

or
$$\frac{\sin n\theta}{\sin \theta} = (2 \cos \theta)^{n-1} - (n-2)_1 (2 \cos \theta)^{n-3} \\ + (n-3)_2 (2 \cos \theta)^{n-5} - \dots \\ + (-1)^r (n-r-1)_r (2 \cos \theta)^{n-2r-1} + \dots$$

3. Prove that

$$2 \cos n\theta = \frac{\sin (n+1)\theta}{\sin \theta} - \frac{\sin (n-1)\theta}{\sin \theta},$$

and hence that

$$2 \cos n\theta = (2 \cos \theta)^n - n(2 \cos \theta)^{n-2} + \frac{n(n-3)}{2!} (2 \cos \theta)^{n-4} - \dots \\ + (-1)^r \frac{n(n-r-1)(n-r-2)\dots(n-2r+1)}{r!} (2 \cos \theta)^{n-2r} + \dots$$

4. If $\cos n\theta$ be given, show that this equation in $\cos \theta$ is satisfied by

$$\cos \theta, \cos \left(\theta + \frac{2\pi}{n} \right), \cos \left(\theta + \frac{4\pi}{n} \right), \dots \cos \left(\theta + \overline{n-1} \cdot \frac{2\pi}{n} \right),$$

and hence show that

$$\cos \theta + \cos \left(\theta + \frac{2\pi}{n} \right) + \dots + \cos \left(\theta + \overline{n-1} \cdot \frac{2\pi}{n} \right) = 0,$$

and
$$\cos^2 \theta + \cos^2 \left(\theta + \frac{2\pi}{n} \right) + \dots + \cos^2 \left(\theta + \overline{n-1} \cdot \frac{2\pi}{n} \right) = \frac{n}{2}.$$

5. Find the last term of the series in No. 3, according as n is even or odd.

6. Find the last term but one of the series in No. 3, according as n is even or odd.

7. Prove that

$$\sec \theta + \sec \left(\theta + \frac{2\pi}{n} \right) + \sec \left(\theta + \frac{4\pi}{n} \right) + \dots + \sec \left\{ \theta + (n-1) \frac{2\pi}{n} \right\}$$

is equal to 0 or $(-1)^{\frac{n-1}{2}} n \sec n\theta$, according as n is even or odd.

8. Prove that

$$\sec^2 \theta + \sec^2 \left(\theta + \frac{2\pi}{n} \right) + \dots + \sec^2 \left\{ \theta + (n-1) \frac{2\pi}{n} \right\}$$

is equal to $\frac{n^2}{1 - (-1)^{\frac{n}{2}} \cos n\theta}$ or to $n^2 \sec^2 n\theta$, according as n is even or odd.

95.

POSITION OF A CURVE WITH REGARD TO ITS ASYMPTOTES.

1. Determine the position of the curve

$$y = \frac{(x-2)(x-4)}{(x-3)(x-5)}$$

with regard to its asymptotes, by considering the sign of the fraction for different values of x .

2. Determine the position of the curve

$$y^2 = \frac{(x-2)(x-4)}{(x-3)(x-5)}$$

with regard to its asymptotes, by considering the sign of the fraction for different values of x .

3. Determine the position of the curve

$$c^2 y^2 - b^2 x^2 = -a^2 b^2$$

with regard to its asymptotes, by expanding y in a series of descending powers of x .

4. Determine the position of the curve

$$xy(x+y)=x+y+1$$

with regard to its asymptotes, by solving for x in terms of y .

5. Determine the position of the curve

$$x^3+y^3=3axy$$

with regard to its asymptotes, by showing that the curve must lie entirely on the positive side of the asymptote.

6. Determine the position of the curve

$$x^3-y^3+a^2(x+y)=0$$

with regard to its asymptote, by expressing the equation of the curve in the form

$$y=x+\frac{\lambda}{x}+\dots,$$

and equating the coefficient of x to zero.

7. Determine the position of the curve

$$x^2y+a^2x=y^3$$

with regard to its asymptotes $y=0$, $y=x$, $y=-x$, by substituting

$$y=\frac{\lambda}{x}, \quad y=x+\frac{\lambda}{x} \quad \text{and} \quad y=-x+\frac{\lambda}{x}$$

in the equation of the curve.

8. Determine the position of the curve

$$x^2y-x^3=a^3$$

with regard to its asymptote, by substituting

$$y=x+\frac{\lambda}{x}+\frac{\mu}{x^2}$$

in the equation of the curve.

96.

THE SEMI-CUBICAL PARABOLA.

1. Show that the curve $ay^2 = x^3$ (i) is symmetrical with respect to the axis of x , (ii) lies entirely on the positive side of the axis of y , and (iii) touches the axis of x at the origin.

2. Show that the curve has a cusp of the first order in the neighbourhood of the origin, and trace the curve.

3. Find the equation of the normal at the point (am^2, am^3) of the curve $ay^2 = x^3$, and the values of m for which the normal passes through the point $(\frac{a}{2}, 0)$.

4. A variable tangent to the curve $ay^2 = x^3$ meets the axes in L and M , and perpendiculars to the axes at L and M meet in P . Prove that the locus of P is a curve similar to the original curve and $4/27$ of the linear dimensions.

5. Show that the tangent at (x, y) to the curve $ay^2 = x^3$ intersects the curve again at the point $(\frac{x}{4}, -\frac{y}{8})$.

6. The tangent at any point P of the curve $ay^2 = x^3$ meets the curve again in Q ; prove that

$$\tan xOP = 2 \cdot \tan xOQ.$$

7. The tangent at P to the curve $ay^2 = x^3$ meets it again at Q ; prove that the locus of the point which divides PQ in the ratio $2 : 1$ is the curve $2ay^2 = x^3$.

8. Prove that two tangents to the curve $ay^2 = x^3$ are also normals, and find their equations.

97.

THE CARDIOIDE.

1. Defining a *cardioide* as the locus of the foot of the perpendicular drawn from a fixed point on a circle to a variable tangent, show that the equation of the curve is

$$r = a(1 + \cos \theta) \quad \text{or} \quad r = a(1 - \cos \theta),$$

according as the initial line passes through the centre or, when produced backwards, passes through the centre.

2. In the cardioide $r = a(1 - \cos \theta)$, show that $\phi = \frac{1}{2}\theta$, and, hence, that the tangents at the ends of a chord drawn through the pole are at right angles.

3. For the cardioide $r = a(1 - \cos \theta)$, prove that

$$(i) \ p = 2a \sin^3 \frac{\theta}{2}, \quad (ii) \ 2ap^2 = r^3.$$

4. If three parallel tangents be drawn to the cardioide $r = a(1 - \cos \theta)$, show that the radii vectores of the points of contact are equally inclined to one another.

5. If the tangents at the points P, Q, R on the curve $r = a(1 - \cos \theta)$ be parallel, prove that of the quantities $\sqrt{OP}, \sqrt{OQ}, \sqrt{OR}$, the sum of two will be equal to the third, where O is the origin.

6. Prove that, if the normals to $r = a(1 - \cos \theta)$ at the points α, β, γ be concurrent,

$$\tan \frac{\beta}{2} \tan \frac{\gamma}{2} + \tan \frac{\gamma}{2} \tan \frac{\alpha}{2} + \tan \frac{\alpha}{2} \tan \frac{\beta}{2} + 3 = 0.$$

7. Find the radius of curvature at any point of a cardioide, by means of the equation (i) $r = a(1 - \cos \theta)$, (ii) $2ap^2 = r^3$, and show that the length of the chord of curvature through the pole is $4r/3$.

8. Show also that the cardioide is (i) the envelope of circles drawn on radii vectores of the given circle as diameters, (ii) the inverse of a parabola, the centre of inversion being at the vertex, (iii) the locus of a point on the circumference of a circle which rolls without slipping on the circumference of an equal circle.

98.

THE FOLIUM OF DESCARTES.

1. Show that the folium

$$x^3 + y^3 = 3axy$$

is symmetrical with respect to the line $y=x$, and find the coordinates of the points in which this line cuts the curve.

2. Find the equations of the tangents at the origin and of the asymptote.

3. Determine the position of the curve with regard to its asymptote and trace the curve.

4. Find the equation of the tangent at any point of the curve.

5. If the intercept which the tangent makes on the axis of y be equal to $-a$, prove that $x=2y$ at the point of contact of the tangent.

6. Show that the points of contact of tangents from the point (h, k) lie on the curve

$$hx^2 - axy + ky^2 = a(kx + hy).$$

7. Find the radii of curvature of the curve at the origin.

8. Show that the coordinates of the centre of curvature at the point where $x=y$ are

$$\left(\frac{21a}{16}, \frac{21a}{16}\right).$$

99.

THE CYCLOID.

Definitions.—If a circle roll without slipping in one plane along a straight line, any point on its circumference traces a *cycloid*.

Let C, D be the points where the tracing point P meets the straight line on which the circle rolls, A the point on the curve farthest from CD , AB the corresponding diameter of

the rolling or *generating* circle, so that AB is perpendicular to CD . AB is called the *axis* of the curve, CD the *base*, A the *vertex*, and C, D the *cusps*.

1. If RPS be the generating circle in any position and P the tracing point, show that

(i) $SC = \text{arc } SP$, (ii) CB and BD are each half the circumference of the circle, and (iii) $AR = \text{arc } PR$.

2. Show that, as the circle RPS is turning about the point P , PR is a tangent to the cycloid. Hence, show that the tangent at the vertex is parallel, and the tangent at a cusp perpendicular, to the base.

3. If the axis of the cycloid be taken as the axis of x and the tangent at the vertex as axis of y , show that

$$x = a(1 - \cos \theta), \quad y = a(\theta + \sin \theta),$$

where a is the radius of the generating circle, and θ the angle ROP through which the radius to the tracing point has turned from the vertex to the position P .

4. Show that the radius of curvature at the point θ of the cycloid

$$x = a(1 - \cos \theta), \quad y = a(\theta + \sin \theta)$$

is $4a \cos \frac{\theta}{2}$ or twice PS .

5. Find the latus rectum of the closest parabola that can be drawn to a given cycloid at the vertex.

6. If the circle turn through the small angle POp , show that the centre moves through a distance Pp , the motion of the generating point P is the resultant of Pp due to rotation, and pP' (equal to Pp and parallel to the base) due to the translation of the centre of the circle, and that PP' coincides ultimately with PR . Hence, show that, ultimately,

$$PP' = 2(RP - Rp),$$

and thus that $\text{arc } AP = \text{twice chord } RP$,

and that the length of the arc of the cycloid from cusp to cusp is four times the diameter of the generating circle.

7. Show that the evolute of a given cycloid is an equal cycloid.

8. If P, P' be two neighbouring points on a cycloid, Q, Q' the corresponding points on the generating circle on AB as diameter, p, p' the corresponding points on the evolute, S, S' the intersections of the normals at P, P' with the base, H, K the intersections of $P'p', Q'B$ with PQ , show that

$$\triangle p'PH = 4 \cdot \triangle BQK,$$

and thus that trap. $PSS'H=3.\triangle BQK$, and hence that the area of the cycloid from cusp to cusp is three times that of the generating circle.

100.

SOME STANDARD FORMS OF INTEGRALS.

1. By putting $\tan \frac{x}{2} = y$, show that

$$\int \frac{dx}{\sin x} = \int \frac{dy}{y} = \log \tan \frac{x}{2}.$$

Hence, by putting $\frac{\pi}{2} + x$ for x , show that

$$\int \frac{dx}{\cos x} = \log \tan \left(\frac{\pi}{4} + \frac{x}{2} \right).$$

2. By putting $x = a \sin \theta$, show that

$$\begin{aligned} \sqrt{(a^2 - x^2)} dx &= \frac{a^2}{2} \int (1 + \cos 2\theta) d\theta \\ &= \frac{1}{2} x \sqrt{(a^2 - x^2)} + \frac{1}{2} a^2 \sin^{-1} \frac{x}{a}. \end{aligned}$$

3. By putting $x = a \tan \theta$, show that

$$\begin{aligned} \int \frac{dx}{\sqrt{(x^2 + a^2)}} &= \int \frac{d\theta}{\cos \theta} = \log (\tan \theta + \sec \theta) \\ &= \log (x + \sqrt{x^2 + a^2}). \end{aligned}$$

4. Integrating by parts, show that

$$\int \sqrt{(x^2 + a^2)} dx = x \sqrt{(x^2 + a^2)} - \int \sqrt{(x^2 + a^2)} dx + a^2 \int \frac{dx}{\sqrt{(x^2 + a^2)}},$$

and therefore that

$$\int \sqrt{(x^2 + a^2)} dx = \frac{1}{2} \{ x \sqrt{(x^2 + a^2)} + a^2 \log (x + \sqrt{x^2 + a^2}) \}.$$

5. By putting $x = a \sec \theta$, show that

$$\int \frac{dx}{\sqrt{(x^2 - a^2)}} = \int \frac{d\theta}{\cos \theta} = \log (x + \sqrt{x^2 - a^2}).$$

6. Integrating by parts, show (as in No. 4) that

$$\int \sqrt{(x^2 - a^2)} dx = \frac{1}{2} \{ x \sqrt{(x^2 - a^2)} + x^2 \log (x + \sqrt{x^2 - a^2}) \}.$$

7. Show that

$$\int \frac{dx}{\sqrt{(ax^2 + 2bx + c)}} = \frac{1}{\sqrt{a}} \log \{ ax + b + \sqrt{a(ax^2 + 2bx + c)} \}.$$

8. Show that

$$\int \frac{dx}{a + b \cos x} = \int \frac{\sec^2 \frac{x}{2} dx}{(a + b) + (a - b) \tan^2 \frac{x}{2}},$$

and thence, by putting $\tan \frac{x}{2} = y$, that the value of the integral is

$$\frac{2}{\sqrt{(a^2 - b^2)}} \tan^{-1} \left(\sqrt{\frac{a-b}{a+b}} \tan \frac{x}{2} \right)$$

or

$$\frac{1}{\sqrt{(b^2 - a^2)}} \log \frac{\sqrt{b+a} + \sqrt{b-a} \cdot \tan \frac{x}{2}}{\sqrt{b+a} - \sqrt{b-a} \cdot \tan \frac{x}{2}},$$

according as a is greater or less than b .

PART II.

101.

Give an account of the process by which the highest common factor of two polynomials $f(x)$ and $\phi(x)$ is obtained, and indicate the principles on which the process depends.

Prove that, if X_1, X_2, X_3 are the functions of x used as divisors at three successive stages of the process, a value of x which makes X_2 vanish will make $X_1 = X_3$. (Trin. 1913.)

102.

Discuss completely the range of values possible for each of the three following functions when the variable x admits of any real value, and the other letters denote real constants :

$$(i) \ ax^2 + 2bx + c. \quad (ii) \ \frac{Ax + B}{x^2 + \alpha^2}. \quad (iii) \ \frac{A}{x - \alpha} + \frac{B}{x + \beta} \ (a \neq \beta).$$

(Pemb. 1909.)

103.

Consider the values taken by the expression

$$\frac{ax^2 + 2bx + c}{a'x^2 + 2b'x + c'}$$

for all real values of x , discriminating between the cases of real or imaginary roots of the two quadratics ; and show graphically or otherwise that the expression always has two stationary values, except in one case of relative distribution of the roots.

(Trin. 1907.)

104.

Discuss the expression of a rational function of x as the sum of a polynomial and of partial fractions whose denominators are powers of linear or quadratic functions of x and numerators respectively constants or linear functions of x .

Illustrate by the cases

$$\begin{aligned} \text{(i)} \quad & \frac{x^3}{(x+1)(x+2)} \cdot & \text{(ii)} \quad & \frac{x^2}{(x^2+x+1)(x+1)} \cdot \\ \text{(iii)} \quad & \frac{x^4}{(x^2+1)^2(x+1)^3} \cdot & & \text{(Pemb. 1913.)} \end{aligned}$$

105.

Enunciate and prove some of the simple tests of the convergence of an infinite series, and apply the tests to the series

$$\begin{aligned} \text{(i)} \quad & 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \dots \\ \text{(ii)} \quad & \frac{a}{b} + \frac{a(a+c)}{b(b+c)}x + \frac{a(a+c)(a+2c)}{b(b+c)(b+2c)}x^2 + \dots \end{aligned}$$

(Trin. 1905.)

106.

State and prove the leading properties in the theory of determinants, and indicate some applications of the theory.

(Trin. 1910.)

107.

A tetrahedron has each edge perpendicular to the opposite edge. Prove that the four perpendiculars from the vertices on the opposite faces are concurrent, the perpendicular from each vertex on the opposite face passes through the orthocentre of that face, and the sum of the squares of opposite edges is the same for each of the three pairs.

(Pemb. 1913.)

108.

The upper face of a layer of rock below ground is plane ; this plane is determined by drilling vertical holes of lengths p, q, r at points A, B, C of the ground, which is horizontal. Find the line in which the rock surface meets the plane ABC ; and show that the inclination θ of the rock surface to the horizontal is given by

$$2\Delta \tan \theta = \{a^2(p-q)(p-r) + b^2(q-r)(q-p) + c^2(r-p)(r-q)\}^{\frac{1}{2}},$$

where Δ is the area, and a, b, c the lengths of the sides of the triangle ABC . (Pemb. 1912.)

109.

Give an account of the principal properties of a system of coaxal circles. (Pemb. 1907.)

110.

Discuss the Principle of Proportional Parts as applied to Mathematical Tables. Investigate in what parts of the tables, and in what manner, the principle fails in the cases of seven-figure tables of (a) natural sines, (b) logarithmic sines, (c) natural tangents, (d) logarithmic tangents. (Trin. 1911.)

111.

The equation of a conic is

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0,$$

where a, b, c, f, g, h have given numerical values. Show how to determine the lengths of its axes, its eccentricity, the coordinates of its centre and foci, and the equations of its axes, asymptotes, and directrices. Consider in particular the conic

$$x^2 + 4xy + y^2 - 2x - 6y = 0. \quad (\text{Trin. 1908.})$$

112.

Find the equations of the curves defined from the following properties :

(1) The locus of a point which moves so that the sum of its distances from two fixed points is constant.

(2) The locus of a point whose distance from a fixed point bears a constant ratio (less than unity) to its perpendicular distance from a fixed line.

(3) The section of a circular cylinder by any plane.

(4) The curve traced by a point of a rod whose ends slide on two fixed perpendicular lines.

Show that these curves are all of the same kind, and determine for each the conditions that it should be an ellipse of major axis 10 inches and minor axis 6 inches. (Pemb. 1908.)

113.

Starting from the definition of a differential coefficient, develop methods and results which will enable you to differentiate any function obtained by combining exponential functions, circular functions, powers, and the inverses of these functions. (Trin. 1910.)

114.

Give an account of the proof of Taylor's theorem, with Lagrange's remainder, for the expansion of a function $f(a+x)$, in powers of x . Apply the theorem to obtain the expansions of $(1+x)^n$ and e^x , finding for what values of x these expansions are valid when continued indefinitely. (Pemb. 1909.)

115.

Give a general account of the motion of a projectile *in vacuo* under gravity, dealing in particular with the determination

of the path or paths through a given point Q , when the speed at the point of projection P is given (and equal to $\sqrt{2ga}$); and of the envelope of the paths obtained by varying the initial direction of projection. Give geometrical constructions when suitable. (Pemb. 1908.)

116.

State and prove the chief dynamical theorems respecting Energy and Momentum in the most general form known to you. Give some simple examples of these theorems; and also point out some important natural phenomena which exemplify them. (Trin. 1907.)

117.

Explain the various graphical methods of representing by ordinates and areas the relations between the quantities occurring in Kinematics and Dynamics. Solve by these methods the following question:

The speed of an electric tram is reduced in 10 seconds from 30 miles an hour to 15 miles an hour by the action of a brake which produced a corresponding retardation, the speeds at the end of each second being given in the following table:

t	0	1	2	3	4	5	6	7	8	9	10
v	30	27.3	25	23.1	21.4	20	18.8	17.6	16.7	15.8	15

Draw a time-velocity diagram, and from it deduce: (1) that the tram will have travelled about 305 feet, (2) that the retardation is approximately proportional to the square of the velocity.

Show further, assuming that the retardation continues to diminish in accordance with the above approximate law, that the speed will be reduced to 5 miles per hour in a further 40 seconds, during which interval the tram will have travelled a further 483 feet. (Trin. 1909.)

118.

Give an account of the principle and properties of the Hodograph, and of their application to simple cases of motion.
(Pemb. 1909.)

119.

Establish the theorems with regard to couples which justify the representation of a couple by a vector, including the law for the composition of two couples acting on a rigid body.
(Pemb. 1908.)

120.

Give an outline of the proof of the principle of virtual work as applied to the equilibrium of (1) a single rigid body, and (2) a system of rigid bodies, indicating clearly circumstances under which the virtual work of reactions between bodies may be ignored.

Indicate the kind of problems for which the method is especially suitable, and illustrate by solving some problem.
(Pemb. 1907.)

121.

The equations

$$ax^2 + 2bx + c = 0, \quad Ax^3 + 3Bx^2 + 3Cx + D = 0$$

have at least one root in common. Draw up a table to show the various ways in which the roots (whether equal or unequal) of the quadratic may or may not coincide with the roots of the cubic; and obtain the conditions for the occurrence of each case in terms of the coefficients.
(Trin. 1909.)

122.

Show that, if m, n, a, b be real and $m \neq n, a \neq b$, the expression

$$\frac{m^2}{x-a} - \frac{n^2}{x-b}$$

is such that (1) there are two real values between which it cannot lie for real values of x , and (2) the imaginary values of x which make the expression real and intermediate between these two real values are all included in

$$\frac{bm^2 - an^2}{m^2 - n^2} + \frac{(a-b)mn}{m^2 - n^2}(\cos \theta + i \sin \theta),$$

where θ is any angle not zero or a multiple of π .

(Pemb. 1913.)

123.

Explain what is meant by similar figures in plane geometry and when two such figures are similarly situated. Show that two similar figures in the same plane, whether similarly situated or not, have a centre of similitude O , such that if P, P' and Q, Q' are two pairs of corresponding points, the triangles $OPQ, OP'Q'$ are similar. Give a construction for finding O , considering separately the cases of directly and inversely similar figures.

In particular, if a plane figure varies continuously, so as to be always similar to its initial form, and so that three straight lines of the figure pass through three given points, prove that every other straight line of the figure passes through a fixed point; and that the locus of every point in the figure is a circle.

(Pemb. 1908.)

124.

Show that the necessary and sufficient condition that the three pairs of points A, A' ; B, B' ; C, C' on a straight line should belong to an involution is

$$BC' \cdot CA' \cdot AB' + B'C \cdot C'A \cdot A'B = 0.$$

State the corresponding condition that three pairs of lines through a point should belong to an involution pencil, and

prove that if two of the pairs are at right angles, the third pair are also at right angles. (Pemb. 1913.)

125.

The Theory of Poles and Polars. State various definitions which can be given of the polar of a point with respect to a conic, and prove their equivalence; prove (1) theorems of reciprocity, (2) that the cross ratio of a range of four points is equal to that of the corresponding pencil of polars, (3) that the intersections of a conic with a pair of its conjugate lines form a harmonic range on the conic: apply the theory to investigate properties of the centre and foci of a conic; and prove the general theorem on which depends the theory of duality (reciprocation) as applied to conics. (Trin. 1909.)

126.

Prove the following propositions:

(1) A given conic can be projected into a conic, and at the same time into a given line at infinity.

(2) A given conic can be projected into a circle and a given point into its centre.

(3) Any conic can be so projected that two given points project into a pair of foci.

(4) Any two conics can be projected into confocal conics.

(5) A system of conics touching four straight lines can be projected into a system of confocal conics.

And generalise by projection the following theorems:

(1) Confocal conics cut orthogonally.

(2) Pairs of tangents from any point to a system of confocal conics have the same bisectors.

(3) Given a focus and two tangents to a conic, the locus of the other focus is a straight line. (Trin. 1907.)

127.

Prove that if the straight lines joining corresponding vertices of two triangles in the same plane meet in a point, then the triangles may be regarded (in an infinite number of ways) as the projections, from two points in space, of another triangle, which does not lie in the plane of the original triangles. Hence, or otherwise, show that the corresponding sides of the two original triangles intersect in three collinear points.

(Pemb. 1909.)

128.

Sketch the figures obtained by inverting the following diagrams: (i) paper ruled in squares with a circle inscribed in each square, (ii) a tessellated pavement made of equilateral triangles with a circle inscribed in each triangle. State some of the more immediate properties which should be apparent in the new figures.

(Pemb. 1911.)

129.

State and prove Pascal's and Brianchon's theorems. Discuss various limiting cases in which one or more pairs of the six points or of the six lines coincide.

(Trin. 1913.)

130.

Give an account of the theory of reciprocation with respect to a conic. Illustrate by reciprocation the following theorems:

(i) Pascal's theorem;

(ii) The line joining a pair of corresponding points of two homographic sets of points on a conic envelops a second conic having double contact with the first.

(Trin. 1911.)

131.

Prove that if u is a complex number and m and n are positive integers prime to one another, $u^{m/n}$ has n values. Illustrate by a diagram.

Show that, if

$$r^2 = a^2 + b^2 + c^2 \quad \text{and} \quad \frac{b + ic}{r + a} = z,$$

$$\text{then} \quad \frac{c + ia}{r + b} = \frac{i(1 - z)}{1 + z} \quad \text{and} \quad \frac{a + ib}{r + c} = \frac{1 + iz}{1 - iz}. \quad (\text{Trin. 1913.})$$

132.

Discuss the geometrical representation of the quantity $x + iy$, where x and y are real numbers and $i = \sqrt{-1}$.

Proceed to give geometrical constructions for the addition and multiplication of such quantities; and to show further that, when n is a positive integer,

$$(\cos \theta + i \sin \theta)^n = \cos n\theta + i \sin n\theta.$$

Extend the last theorem to the cases when n is a negative integer, and a fraction.

Show that all the roots of the equation

$$x^n = \cos n\theta + i \sin n\theta$$

can be thus determined, and hence obtain the real quadratic factors of the expression

$$x^{2n} - 2x^n \cos n\phi + 1. \quad (\text{Trin. 1905.})$$

133.

Assuming that $E(z)$ stands for $1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \dots$, show that $E(z) = e^x(\cos y + i \sin y)$, where $z = x + iy$.

Hence, show that $E(z)$ is periodic with period $2\pi i$. Find the roots of $E(z) = 1$, and prove that $E(\frac{1}{2}\pi i) = \pm i$. Show further how to build up the theory of Trigonometric functions of z .

(Trin. 1906.)

134.

Write a short account of the theory of the functions

$$\log z, \exp z$$

(where z is complex), starting from whatever definition of the logarithmic or exponential function you prefer. Prove, in the course of your remarks, that if

$$z = x + iy = r(\cos \theta + i \sin \theta),$$

then $\log z = \log r + (\theta + 2m\pi)i$, $\exp z = e^x(\cos y + i \sin y)$, and obtain the series for $\cos y$ and $\sin y$ in powers of y .

(Trin. 1908.)

135.

If the sine and cosine of a real angle are defined by the infinite series

$$\sin \theta = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \dots, \quad \cos \theta = 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \dots,$$

and the other trigonometrical functions by $\tan \theta = \sin \theta / \cos \theta$, etc., give an outline of the way in which the chief propositions of trigonometry can be established.

Indicate any propositions the proofs of which would be materially easier or harder than in the usual treatment.

[The properties of the exponential functions may be assumed.]

(Trin. 1912.)

136.

Give definitions of $\sin z$ and of the other trigonometrical functions which are applicable when z is a complex number. Sketch the processes by which the usual formulae of analytical trigonometry are established for these functions; and discuss whether all the formulae proved for real angles remain true for the trigonometrical functions of a complex number.

(Pemb. 1908.)

137.

Give a sketch of the processes by which the meaning and fundamental properties of a^x are determined in the cases when x is (1) a positive integer, (2) a positive fraction, (3) a negative integer or fraction, (4) a complex number of the form $\xi + i\eta$, a being in all cases real and positive.

(Pemb. 1907.)

138.

Write down the most general formulae for transformation of coordinates from one rectangular set to another.

Supposing that these formulae change

$$ax^2 + by^2 + c + 2fy + 2gx + 2hxy$$

into $a'x'^2 + b'y'^2 + c' + 2f'y' + 2g'x' + 2h'x'y'$,

prove that $a' + b' = a + b$, $a'b' - h'^2 = ab - h^2$

and $a'b'c' + 2f'g'h' - a'f'^2 - b'g'^2 - c'h'^2$
 $= abc + 2fgh - af^2 - bg^2 - ch^2$,

and give geometrical meanings of the vanishing of each of these quantities.

(Pemb. 1909.)

139.

Show how to reduce the general equation of a conic

$$S \equiv ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$$

to simpler forms.

Consider the various types of simplification that are possible, and the discrimination of cases which arise for central, non-central and degenerate conics.

Obtain equations for determining the coordinates of the foci.

Explain the connection between the point equation and the tangential equation of a conic, and the ideas of polar reciprocation.

Show that, if $\Sigma = Al^2 + Bm^2 + Cn^2 + 2Fmn + 2Gnl + 2Hlm = 0$ be the tangential equation of a conic, the tangential coordinates of the directrices satisfy the equations

$$\frac{l^2 - m^2}{a - b} = \frac{lm}{h} = \frac{\Sigma}{\Delta}. \quad (\text{Trin. 1907.})$$

140.

Determine the different kinds of conics represented by the equation $x^2 + 4\lambda xy + 4y^2 + 2(1 + \lambda)x + 8y + 5 + 2\lambda = 0$ as λ changes from a large positive value to a large negative value.

Examine in particular the critical cases $\lambda = 1, 0, -1, -2$, and illustrate by rough diagrams the transition from one kind of conic to another. (Pemb. 1911.)

141.

The tangents to an ellipse at A and B , extremities of the major and minor axes, meet in K . Show that the osculating circles at A and B have K as an external limiting point, and subtend supplementary angles at K .

Find any such ellipse in relation to the circles

$$\begin{aligned} (x^2 + y^2) \cos^2 \alpha + 2cx + c^2 &= 0, \\ (x^2 + y^2) \sin^2 \alpha + 2cx + c^2 &= 0. \end{aligned} \quad (\text{Pemb. 1910.})$$

142.

A point P moves in such a way that

$$VP - SP = SK,$$

where S is a fixed point in the plane, V a fixed point not in the plane, and K the foot of the perpendicular from V on the plane. Prove that the locus of P is a parabola with focus at S ; and that, if the vertex of this parabola is A , the point V lies on a second parabola with vertex at S and focus at A .

(Pemb. 1910.)

143.

Discuss the number of conics which pass through m given points and touch n given lines in the several cases when m and n are zeros or positive integers, such that $m+n=5$.

(Pemb. 1913.)

144.

Investigate Taylor's Theorem

$$f(x+h) = f(x) + hf'(x) + \frac{h^2}{2!}f''(x) + \dots \\ + \frac{h^{n-1}}{(n-1)!}f^{(n-1)}(x) + \frac{h^n}{n!}f^{(n)}(x+\theta h),$$

stating carefully what assumptions your proof involves as to the nature of $f(x)$. Apply the theorem to establish the exponential, sine and cosine expansions.

Explain the difficulty which arises when we try to apply Taylor's Theorem, with the above (Lagrange's) form of the "remainder," to the logarithmic and binomial expansions, and show how to obtain some other formula for the "remainder" suitable for use in these cases.

(Trin. 1909.)

145.

Explain how the asymptotes of an algebraic curve may be determined, paying attention to the cases in which the curve possesses (i) asymptotes parallel to an axis, (ii) pairs of parallel asymptotes.

Illustrate your answer by considering the curve

$$xy(x-y)^2 - x^3 + y^3 + 2xy = 0. \quad (\text{Pemb. 1912.})$$

146.

Establish formulae for the curvature at any point of a plane curve, including the cases when the curve is defined

(a) by a Cartesian equation, (b) by equations of the type $x=\phi(t)$, $y=\psi(t)$, (c) by an intrinsic equation, (d) by a p, r equation, and (e) as the envelope of a line which contains one variable parameter. Apply such of these formulae as are suitable to the cases of the ellipse and the parabola.

(Trin. 1910.)

147.

Give a definition of the curvature of a plane curve at a point, and find an analytical expression for it when the curve is $f(x, y)=0$. Discuss the cases in which the point is a node or cusp, with special reference to the curves

$$y^3=x(x+y), \quad y^3=(x+y)^2.$$

What points on a curve will correspond to singular points on the evolute ?

(Trin. 1905.)

148.

Show that the cardioide whose equation is $r=2a(1-\cos \theta)$ is (a) the inverse of a parabola, (β) the locus of the extremities of a line of length $4a$ which passes through a fixed point on a circle of radius a and has its middle point on the circumference of that circle, (γ) the curve described by a given point of a circle of radius a which rolls on the exterior of an equal circle.

Deduce some geometrical properties of the curve, and prove that the normals at the extremities of any chord through the pole will meet on a circle of radius a .

(Trin. 1906.)

149.

Write a short essay on the theory of maxima and minima, considering the cases of (a) functions of one variable, (b) functions of several independent variables, (c) functions of two or three variables connected by one or two relations, and

using purely analytical methods as far as possible. Illustrate your remarks by determining the axes of the conic

$$ax^2 + 2hxy + by^2 = 1,$$

(α) by finding the maxima and minima of r , when

$$1/r^2 = a \cos^2 \theta + 2h \cos \theta \sin \theta + b \sin^2 \theta,$$

and (β) by finding the maxima and minima of $x^2 + y^2$, when

$$ax^2 + 2hxy + by^2 = 1. \quad (\text{Trin. 1908.})$$

150.

Determine the maximum and minimum values of the functions

$$(i) \cos x + \frac{1}{2} \cos 2x + \frac{1}{3} \cos 3x,$$

$$(ii) \sin x + \frac{1}{2} \sin 2x + \frac{1}{3} \sin 3x,$$

corresponding to values of x which lie between 0 and 2π .

Draw rough graphs to show the variations in these functions.

(Pemb. 1912.)

151.

Show that the function $\sin x + a \sin 3x$ for values of x from 0 to π has no zeros except the terminal ones if $-\frac{1}{3} < a < 1$. Show also that it has two minima with an intermediate maximum if $a < -\frac{1}{3}$, one maximum if $-\frac{1}{3} < a < \frac{1}{3}$, two maxima and an intermediate minimum if $a > \frac{1}{3}$.

Indicate roughly the types of the graph of the function in the four cases.

(Pemb. 1910.)

152.

Give an account of Attwood's machine, and obtain a formula for calculating g from the observations.

Find how the apparent value of g would be modified if the observations were made within a cage suspended by a chain passing over a smooth pulley and balanced by a counterpoise whose mass M is equal to that of the cage and its contents.

(Pemb. 1908.)

153.

Explain how to determine the acceleration of gravity by means of (1) Attwood's machine, and (2) a simple pendulum. Compare the advantages of the two methods, and point out the difficulties that would occur in making the observations and the assumptions involved in the establishment of the formulae used.

Obtain the approximate formula $A + B \cos 2\lambda$ for the value of g at a point of the earth's surface in latitude λ , taking account of the earth's rotation and assuming the earth to be a sphere which exerts an attraction of constant magnitude tending towards its centre. (Pemb. 1911.)

154.

Prove the existence of an "instantaneous centre" for the motion of a flat body in its own plane.

In the figure the rods AB , BC , CD are smoothly jointed at B and C , and AB and CD can turn about the fixed points

A and D respectively. The angles BCD and CDA are right angles, and $AB = BC = 4$ feet and $CD = 2$ feet.

If the angular velocity of CD is given, find the angular velocity of AB and of BC when the rods are in the position shown.

If each rod is uniform and weighs 2 lbs. per foot of its length, find by Virtual Work or otherwise the couple acting

on CD which will keep the rods at rest in the position shown with AD horizontal. (Pemb. 1913.)

155.

Prove that any displacement of a plane lamina in its own plane can be effected by the rolling of a curve fixed in the lamina on a curve fixed in the plane; and that if the paths of two points of the lamina be known, the curvature of the path of any other point of the lamina can be determined.

Explain how this theorem may be used to determine whether a given position of equilibrium of a statical system under gravity is stable or unstable, and give an illustration.

(Trin. 1912.)

156.

State Newton's experimental law concerning the impact of two smooth elastic spheres, and find the ratio of the impulses between the spheres during compression and during recovery.

The vectors OA , OB represent the velocities before impact of two spheres of masses M and m , and C is taken in AB such that $M \cdot AC = m \cdot CB$: through C is drawn the plane perpendicular to the line of impact, and the perpendiculars AD , BE on this plane from A and B respectively are produced to A' and B' , so that

$$DA' = e \cdot AD, \quad EB' = e \cdot BE.$$

Prove that OC represents the velocity of the centre of inertia, that OA' , OB' represent the velocities of the spheres after impact, and that the apparent loss of kinetic energy is measured by

$$\frac{1}{2} \frac{Mm}{M+m} (AB^2 - A'B'^2). \quad (\text{Pemb. 1907.})$$

157.

Define simple harmonic motion ; determine the path, velocity and acceleration of a particle, whose motion is compounded of two simple harmonic motions in different directions, of equal periods but unequal amplitudes.

Prove the isochronism of the cycloid under gravity ; show that the projection of the particle on any vertical line moves with simple harmonic motion ; and prove that when the particle starts from rest at the cusp, its acceleration is constant in magnitude and directed towards a point moving uniformly in a horizontal line. (Trin. 1911.)

158.

When a point moves uniformly in a circle, its projection on any diameter is said to execute a simple harmonic motion. From this definition deduce the characteristic properties of simple harmonic motion, and apply your methods to prove the following results :

(i) If a particle slides on a smooth cycloid, whose axis is vertical, and starts from any point of the curve, its motion is simple harmonic.

(ii) If a simple pendulum of length l swings through an arc α on each side of the vertical, the time of a complete oscillation lies between $2\pi\sqrt{l/g}$ and $2\pi\sqrt{la/g \sin \alpha}$. (Trin. 1908.)

159.

Investigate the various theorems concerning the Conservation of Energy and the Conservation of Momentum, under appropriate conditions, for a system of two particles.

Apply your theorems to the solution of the following problem : Two particles of masses m and m' are connected by an elastic string, of natural length l and without mass, and are initially at rest at a distance $\frac{1}{2}l$ from each other at

points A and B respectively. Blows P and Q are applied to the particles in directions perpendicular to AB and towards A respectively. Discuss the subsequent motion.

(Trin. 1910.)

160.

Prove that if two particles of masses m_1, m_2 are moving in a plane, their kinetic energy is

$$\frac{1}{2}(m_1 + m_2)V^2 + \frac{1}{2}\frac{m_1 m_2}{m_1 + m_2}v^2,$$

where V is the velocity of their centre of mass and v is their relative velocity.

A shell of mass $(m_1 + m_2)$ is fired with a velocity whose horizontal and vertical components are U, V , and at the highest point in its path the shell explodes into two fragments m_1, m_2 . The explosion produced an additional kinetic energy E , and the fragments separate in a horizontal direction; show that they strike the ground at a distance apart which is equal to

$$\frac{V}{g} \left\{ 2E \left(\frac{1}{m_1} + \frac{1}{m_2} \right) \right\}^{\frac{1}{2}}. \quad (\text{Pemb. 1910.})$$

161.

A, B are two equal and equally rough weights lying on a rough table and connected by a string. A string is attached to B and is pulled in a horizontal direction making an angle α with AB produced until motion is about to ensue. Examine the cases where one or both of the weights are on the point of motion, showing that they arise according as $\alpha \gtrless \frac{\pi}{4}$. Show that in the latter case B is about to move in a direction making an angle 2α with AB produced. Find the force needed in each case.

(Pemb. 1911.)

162.

Prove that the normal component of the acceleration of a point describing a plane curve in any manner is equal to the product of the speed of the point and the angular velocity of the tangent.

A heavy particle P slides on a smooth curve of any form in a vertical plane. The centre of curvature at P is Q , and R is on the same vertical as Q and at the level of zero velocity. Show that the acceleration makes with the normal an angle

$$\tan^{-1}(\tfrac{1}{2} \tan PRQ). \quad (\text{Pemb. 1912.})$$

163.

Prove that, when a particle describes a path under the action of a force directed to a fixed point, the radius vector drawn from the point to the particle describes equal areas in equal times.

A particle of mass m is held on a smooth table. A string attached to this particle passes through a hole in the table and supports a particle of mass $3m$. Motion is started by the particle on the table being projected with velocity V at right angles to the string. If a is the original length of the string on the table, prove that when the hanging weight has descended a distance $a/2$ (assuming this possible) its velocity will be

$$\frac{\sqrt{3}}{2} \sqrt{ga - V^2}. \quad (\text{Pemb. 1913.})$$

164.

State the principle of Virtual Work. Prove it (1) for forces acting at a point, (2) for forces acting on a system of connected particles; and extend it to the case of a rigid body under the action of given forces, pointing out the assumptions and limitations of the principle.

134 SUBJECTS FOR MATHEMATICAL ESSAYS

Two corners of a regular pentagon of light freely jointed rods are connected by a string in tension, and equilibrium is maintained by another string also in tension, connecting one corner to the middle of the opposite side, the strings being perpendicular to each other. Show that their tensions are in the ratio $2 \sin \frac{1}{5}\pi : 1$. (Trin. 1913.)

165.

Give a short account of Graphic Methods in Statics, embracing the following points: (1) Enunciation of property of force polygon, (2) Application to determine stresses at smooth joints of systems of rods, (3) Illustration by application to a simple problem where heavy rods form a regular hexagon, (4) Properties of "funicular" diagram, (5) Graphic solution of simple illustrative problems. (Trin. 1905.)

166.

Explain the application of graphical methods to kinematical problems, showing how to determine velocity, space described, and kinetic energy, from a knowledge of the acceleration, and conversely.

A particle starts with given velocity V and moves under a retardation equal to μ times the space described. Show that the distance traversed before it comes to rest is $V/\sqrt{\mu}$. (Trin. 1911.)

167.

In the figure, AB , BC are light rods hinged at B ; A is a hinge attached to a fixed support, and a weight W is hung from C . The system rests over a cylindrical drum which can turn freely round a fixed axis F . Show that the couple required to turn the drum in the direction of the arrow is

$$2\mu W(\cos \alpha + \mu \sin \alpha)(CE^3/CD),$$

where μ is the coefficient of friction between the drum and the rods.
(Pemb. 1912.)

168.

Five weightless rods AB , BC , CD , DA and BD , smoothly jointed at their ends, form a framework; AB is vertical with B above A , $BAD=30^\circ$, $ABD=60^\circ$, BC is horizontal and $BD=DC$. A weight W is suspended from C , and the framework is supported by a horizontal force at B and a force at A . Find the stress in each of the rods, distinguishing between thrust and tension.
(Pemb. 1910.)

169.

A plane polygon is formed of a system of rods smoothly hinged together in succession; show how to determine graphically necessary and sufficient conditions of equilibrium when forces are applied at the joints.

State the number of conditions, and show how to determine the stresses in the rods.
(Pemb. 1909.)

170.

Give an account of the method in which the force polygon and the funicular polygon may be used to find the resultant of a system of forces in a plane ; and illustrate the method by a sketch of the case where forces of 2, 3 and 4 lbs. weight act respectively along AB , BC and CD , three sides of a rectangle $ABCD$.

A string is supported at its two ends and has weights attached to points along it. Show that, when the system is in equilibrium, the form of the string is a possible form of funicular polygon for the weights, and that the closing sides of the funicular polygon lie along the first and last portions of the string. (Pemb. 1907.)

171.

Quantities $\alpha_1, \alpha_2, \alpha_3, \dots$ are formed, depending on the ratio of consecutive terms of the infinite series of real and positive terms $S \equiv u_1 + u_2 + u_3 + \dots$, according to the following laws : $\alpha_1 = u_n / u_{n+1}$; $\alpha_2 = n(\alpha_1 - 1)$; $\alpha_3 = (\alpha_2 - 1) \log n$, and furnish the following tests :

(1) If α_1 is always < 1 , or approaches unity as a limit, *from below*, then S is divergent.

(2) If α_1 is always $> k$, where $k > 1$, then S is convergent.

(3) If α_1 approaches unity as a limit, *from above*, then the case is undecided, and α_2 must be examined, and the same series of tests (1), (2), (3) holds good, and so on.

Prove the theorems up to and including the tests furnished by α_3 . (Trin. 1907.)

172.

Prove that if m is prime to a the least positive remainders of the series of integers

$$k, k+a, \dots k+(m-1)a$$

with respect to m are a permutation of the numbers of the series

$$0, 1, 2, \dots (m-1),$$

and that

$$a^{\phi(m)} - 1 \equiv 0 \pmod{m},$$

where $\phi(m)$ is the number of integers less than m and prime to it.

Prove also that $(m-1)! + 1$ is divisible by m if, and only if, m is a prime.

Show that if m is a prime, and $p < m$,

$$(p-1)!(m-p)! + (-1)^{p-1} \equiv 0 \pmod{m}.$$

(Trin. 1910.)

173.

A polynomial $\phi(x)$ is defined by the identity

$$\phi(x+1) - \phi(x) = (1+x)^r$$

with $\phi(1)=1$, where r is a positive integer. Prove that

$$\phi(0)=0, \quad \phi(-1)=0,$$

and from

$$\phi(-x) - \phi(-x-1) = (-1)^r x^r$$

deduce that

$$\phi(x) = (-1)^{r-1} \phi(-x-1)$$

for all integral and therefore all values of x .

Hence show that the sum of the r th powers of the first n natural numbers is an integral function of n divisible by $n(n+1)$, and that it is also divisible by $(2n+1)$ if r is even.

(Pemb. 1910.)

174.

Establish the harmonic properties of the figures determined by four points and four straight lines, and deduce a construction by the ruler alone for the harmonic conjugate of a given point with respect to two given points collinear with it. Prove that the harmonic (diagonal) points of a quadrangle inscribed in a conic form a triangle self-conjugate with respect to the conic; consider also the reciprocal theorem. Show

that if a quadrilateral be circumscribed to a conic, its harmonic (diagonal) points coincide with those of the quadrangle formed by the points of contact. (Trin. 1912.)

175.

“Two ranges are homographic when the cross ratio of any four points of the one is equal to that of the corresponding four points of the other.”

“Two ranges are homographic when the distance x of a point of one range from a fixed point on its line and the distance x' defined similarly for the other range are connected by a relation

$$axx' + bx + cx' + d = 0."$$

Give a general sketch of the theory of homographic ranges and ranges in involution, showing that the two definitions lead to the same results, and including in your account proofs of the following theorems :

- (i) two homographic ranges are projective ;
- (ii) two homographic ranges can be placed so as to be in perspective ;
- (iii) the lines joining corresponding points of two homographic ranges envelop a conic ;
- (iv) a transversal cuts the pairs of opposite sides of a quadrangle in six points in involution ;
- (v) pairs of points on a line, conjugate with respect to a conic, form an involution. (Trin. 1908.)

176.

Determine the number of different values of $\tan \frac{m\theta}{n}$ which are consistent with a given real value of $\tan \theta$, when m and n are given positive integers prime to each other.

Find the algebraical equation from which the solution of which these values may be determined.

In the case when $\tan \theta$ and $\sin \frac{1}{2}\theta$ have given values, consistent with each other, find how many values are possible for $\tan \frac{\theta}{16}$. (Pemb. 1909.)

177.

Enunciate definitions of the trigonometrical functions, Cosine and Sine; and give a proof of the addition theorem for $\cos(A+B)$, in accordance with your definitions, valid for all values of A and B .

Show without actual determination of the functions that $\cos n\theta$ and $\frac{\sin n\theta}{\sin \theta}$ are rational integral functions of $2 \cdot \cos \theta$ of degrees n and $n-1$ respectively.

Show that if m and n are both odd integers, $m < n$,

$$\frac{\sin m\theta}{\sin n\theta} = \frac{1}{n} \sum_{r=1}^{n-1} (-1)^r \sin \frac{mr\pi}{n} \cot \left(\theta - \frac{r\pi}{n} \right).$$

(Trin. 1907.)

178.

Show how to obtain the formulae

$$2 \cos n\theta = (2 \cos \theta)^n - n(2 \cos \theta)^{n-2} + \frac{n(n-3)}{2!} (2 \cos \theta)^{n-4} - \dots,$$

$$(-1)^{\frac{n}{2}} \cos n\theta = 1 - \frac{n^2}{2!} \cos^2 \theta + \frac{n^2(n^2-1^2)}{4!} \cos^4 \theta - \dots,$$

where n is a positive integer, and in the second formula an even integer. Write down the general terms in each case, and give a formula to replace the second when n is odd. Obtain similar expressions for $(\sin n\theta)/(\sin \theta)$ in terms of $\cos \theta$.

Conversely, show how any polynomial in $\cos \theta$ and $\sin \theta$ may be expressed as a sum of constant multiples of terms of the type $\cos n\theta$, $\sin n\theta$. (Trin. 1909.)

179.

Show how to reduce the general equation of a conic, referred to rectangular axes,

$$S \equiv ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0,$$

to simpler forms.

Consider the various types of simplification that are possible, and the discrimination of cases which arise for central, non-central and degenerate conics.

Show, by considering the tangential equation

$$Al^2 + Bm^2 + Cn^2 + 2Fmn + 2Gnl + 2Hlm = 0$$

of the conic, or otherwise, that the coordinates of the foci are given by the equations

$$Cx^2 - 2Gx + A = \kappa, \quad Cy^2 - 2Fy + B = \kappa,$$

where κ is either root of the quadratic

$$C\kappa^2 - (a+b)\Delta\kappa + \Delta^2 = 0. \quad (\text{Trin. 1913.})$$

180.

Show that the straight line $lx + my + n = 0$, whose coefficients are connected by the relation

$$al^2 + bm^2 + cn^2 + 2fmn + 2gnl + 2hlm = 0,$$

in general envelops a conic.

The axes of coordinates being supposed rectangular, find in terms of a, b, c, f, g, h the conditions that the conic shall be (1) a circle, (2) a parabola, (3) a rectangular hyperbola; and explain how to distinguish between the ellipse and hyperbola.

Determine also the nature of the envelope in the cases when the given expression breaks up into two linear factors, real and distinct, coincident or imaginary. (Pemb. 1908.)

181.

The equation of a conic, referred to rectangular axes, being given in the form

$$S \equiv ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0,$$

show that the equation for its semi-axes is

$$C^3r^4 + C\Delta(a+b)r^2 + \Delta^2 = 0.$$

Obtain the equation of (i) its asymptotes, (ii) its director circle; and show that the equation of its directrices may be written in the form $\Delta S + Cr^2(X^2 + Y^2) = 0$,

where $X \equiv ax + hy + g$, $Y \equiv hx + by + f$ and r is a root of the quadratic above.

Prove that if the equation of an ellipse, referred to areal coordinates, be

$$ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy = 0,$$

the ratio of its area to that of the triangle of reference is

$$2\pi\Delta(A+B+C+2F+2G+2H)^{-\frac{3}{2}}. \quad (\text{Trin. 1911.})$$

182.

Show that the locus of centres of a family of conics through four given points is a conic. Show also that, in the case of a family of conics having three-point contact at P and passing through a fourth point Q , the locus of centres touches the conics at P , has curvature at P of the opposite sign and of double the magnitude of that of the conics, and has PR as a diameter where R is the middle point of PQ .

What is the locus of centres when the family has four-point contact? (Pemb. 1912.)

183.

Show that the equations

$$x : y : 1 = a_1 t^2 + 2b_1 t + c_1 : a_2 t^2 + 2b_2 t + c_2 : a_3 t^2 + 2b_3 t + c_3$$

are equivalent to equations of the form

$$t^2 : 2t : 1 = A_1 x + A_2 y + A_3 : B_1 x + B_2 y + B_3 : C_1 x + C_2 y + C_3.$$

Show also that, if the equations be taken as the definition of a conic, where x and y are rectangular coordinates and t a variable parameter, the existence of foci and directrices is established by the fact that α and β can be chosen to make both the expressions

$$a_1 t^2 + 2b_1 t + c_1 - \alpha(a_3 t^2 + 2b_3 t + c_3) \\ \pm i \{a_2 t^2 + 2b_2 t + c_2 - \beta(a_3 t^2 + 2b_3 t + c_3)\}$$

perfect squares.

Find the focus of the parabola

$$x = t^2 - 2t + 2, \quad y = t^2 + 1. \quad (\text{Pemb. 1913.})$$

184.

$$\text{If} \quad f(\xi) = \phi(\xi), \quad f'(\xi) = \phi'(\xi), \quad \dots \quad f^{(n-1)}(\xi) = \phi^{(n-1)}(\xi),$$

$$\text{but} \quad f^{(n)}(\xi) \neq \phi^{(n)}(\xi),$$

$$\text{the curves} \quad y = f(x), \quad y = \phi(x)$$

have contact of the n th order at the point for which $x = \xi$.

Taking the curve $y = f(x)$ to be (i) any straight line, (ii) any circle, (iii) any conic, discuss the problem of so choosing it as to have contact with the curve $y = \phi(x)$ of as high an order as possible. Determine incidentally the equation of the tangent to a curve, the condition for a point of inflexion and the equation of the circle of curvature. Finally, supposing that $y = f(x)$ touches the axis of x at the origin, determine the general equation of a conic having contact of the third order with the curve there, and the equation of the osculating conic.

(Trin. 1908.)

185.

The radius of curvature ρ , at a point P of a curve, is frequently defined as $\frac{ds}{d\phi}$, where s is the arc of the curve and ϕ the inclination of the tangent to a fixed straight line; it is also defined as the limit of R (as P' approaches P), where R is the radius of the circle which touches the curve at P and intersects the curve again in the neighbouring point P' .

Prove that these two definitions of ρ are consistent for a curve given by the equation $y=f(x)$, provided that $f''(x)$ is not zero at P .

Show also that if P is an inflexion (so that $f''(x)=0$) and ρ' is the radius of curvature at P' , then, as P' approaches P , $\lim(\rho'/R)=\frac{1}{3}$. (Pemb. 1908.)

186.

The equation of a rational algebraic curve of the n th degree being written in the form

$$x^n f_0\left(\frac{y}{x}\right) + x^{n-1} f_1\left(\frac{y}{x}\right) + x^{n-2} f_2\left(\frac{y}{x}\right) + \dots = 0,$$

obtain the line asymptotes of the curve, and discuss the case of the parallel line asymptotes, determining the condition that the curve may have a cusp at infinity.

Show that if

$$f_0(\mu)=0, \quad f_0'(\mu)=0, \quad f_0''(\mu) \neq 0, \quad f_1(\mu) \neq 0,$$

then any of the family of parabolas given by

$$x f_1(\mu) + \frac{1}{2}(y - \mu x)^3 f_0''(\mu) + (y - \mu x) \left\{ f_1'(\mu) - \frac{1}{3} \frac{f_1(\mu) f_0'''(\mu)}{f_0''(\mu)} \right\} = 0$$

has four-point contact with the curve at infinity.

Illustrate by sketching the curve

$$x^2 y^2 - 2x^3 - 2x^2 = xy + x + y. \quad (\text{Trin. 1911.})$$

187.

A curve touches the axis of x at O , and P is a point on it at a small arcual distance s from O . Prove that, neglecting fourth and higher powers of s , the coordinates of P are

$$x = s - \frac{s^3}{6\rho^2}, \quad y = \frac{s^2}{2\rho} - \frac{s^3}{6\rho^2} \frac{d\rho}{ds},$$

where ρ and $\frac{d\rho}{ds}$ are respectively the values at O of the radius of curvature and its differential coefficient with regard to s .

The normals at O and P meet in Q . Prove that to the same degree of approximation the difference between QO and QP is

$$\frac{s^3}{12\rho^2} \frac{d\rho}{ds}. \quad (\text{Pemb. 1911.})$$

188.

Discuss the problem of determining, and discriminating between, the maxima and minima of a function of two or three independent variables. Illustrate your theory by finding the maximum and minimum values of

$$(x^2 + 2xy + 2y^2)e^{-(2x^2 + 3y^2)}. \quad (\text{Trin. 1912.})$$

189.

Write an account of the motion of a particle in a straight line under a restoring force proportional to the displacement from a position of equilibrium; and deduce the period of small oscillations of a simple pendulum of length l .

A heavy particle is fastened to a point of a uniform elastic thread which is stretched between two fixed points A , B in the same vertical line. The particle is let go from rest in such a position that the tensions in the two parts of the thread are initially equal; show that, if the tension is sufficiently great, the oscillation will have the same period as a simple

pendulum of length c , where $2c$ is the vertical distance through which the particle falls before next coming to rest. Express the necessary limit of the initial tension in terms of the weight of the particle and the unstretched lengths a , b of the two parts of the thread. (Pemb. 1912.)

190.

Write an essay on the theory of dimensions of dynamical quantities, indicating (i) the application of the theory to determine the units of the leading dynamical quantities in terms of the three fundamental units of mass, space and time ; (ii) the use of the theory to deduce dynamical results directly from the consideration of the equations, *e.g.* the nature of the period of a simple harmonic motion.

(i) Show that one horse-power $= (746 \times 10^7)$ c.g.s. units of power ; taking 1 lb. $= 453.6$ grammes, $g = 32.2$ f.s. $= 981$ c.s.

(ii) The bending moment at any point of a girder is given as $B \times \text{curvature}$, where B is a quantity, which for girders of the same material and similar cross-sections varies as the square of the cross-section. A weight W is placed at various positions on a horizontal supported girder of weight w and length l , and the (small) deflection (y) at the weight is observed. Show that, if a geometrically similar girder of the same material and length nl be used for similar experiments, the deflection may be calculated from the formula $y' = n^2y$, provided that the weight used be n^3W , and the points of observation divide the girder in the same ratio in the two sets of observations. (Trin. 1912.)

191.

Investigate the various problems arising from the collision of smooth elastic spheres. In particular consider the loss of kinetic energy.

A mass m of water issued per unit time from a pipe with uniform velocity u , and strikes a pail which retains it, there being no elasticity. Initially the pail is at rest, and at a subsequent instant is moving in the direction of the stream with velocity V . Prove that

$$\frac{dU}{dt} = \frac{m(u - V)^3}{Mu^2},$$

and that the loss of energy up to this instant is $\frac{1}{2}MuV$, where M is the mass of the pail, and gravity is omitted from consideration. (Trin. 1910.)

192.

Prove that in the motion of a material system,

(i) the total linear momentum in any direction is that of the whole mass moving with the velocity of the centre of inertia ;

(ii) the kinetic energy of the system is that of the whole mass moving with the velocity of the centre of inertia, together with the kinetic energy of the motion *relative* to the centre of inertia ;

(iii) the centre of inertia moves as a particle of mass equal to that of the system under the action of a force equal to the linear resultant of the forces on the system ;

(iv) the motion relative to the centre of inertia is independent of the motion of that point.

Apply these principles to the following problem. One end of a string of length l is tied to a rod of length a , the other to a fixed point at height h above the ground. The rod is whirled round in a vertical plane, and when it is vertically over the fixed point the string is set free. Show that the rod will be horizontal when it strikes the ground, if its centre had a velocity $\frac{1}{4}(2n+1)\pi(2l+a)\{g/(2h+2l+a)\}^{\frac{1}{2}}$

when set free.

(Trin. 1911.)

193.

Prove that the rate of change of momentum in a fixed direction of a material system is equal to the external force acting on the system in that direction.

Illustrate the theorem by the following example. A particle of mass m moves on a smooth inclined plane of mass M and angle α , which is only free to move in the vertical plane through a line of greatest slope. Determine the pressure of the particle on the plane, and of the plane on the horizontal plane on which it moves; show that if at any time the plane be at rest and the particle have only a horizontal velocity V , the semi-latus rectum of its path on the plane is

$$\frac{V^2}{g} \frac{M + m \sin^2 \alpha}{(M + m) \sin \alpha},$$

that of its path in space is

$$\frac{V^2}{g \sin \alpha} \frac{M + m \sin^2 \alpha}{(M^2 + 2Mm \sin^2 \alpha + m^2 \sin^2 \alpha)^{\frac{1}{2}}};$$

and that of the path in space of the centroid of the system is

$$\frac{V^2 m}{g} \frac{(M + m \sin^2 \alpha)}{(M + m)^2 \sin^2 \alpha}. \quad (\text{Trin. 1912.})$$

194.

State Newton's laws concerning the direct impact of two uniform spheres. Deduce that the impulse during the period of compression bears to the total impulse a ratio which is independent of the masses and velocities of the spheres.

Two smooth spheres of masses M and M' impinge obliquely; lines OP and OQ represent the velocities of M before and after the impact, and lines OP' and OQ' represent the velocities of M' before and after the impact. Prove that PQ and $Q'P'$ are parallel and of lengths inversely proportional to M and M' ; and that, if p, p', q, q' are the projections of P, P', Q, Q'

on any line parallel to PQ , the ratio $q'q : pp'$ is the coefficient of elasticity.

Show that these results are applicable to the case when the velocities of the centres of the spheres before and after impact are not coplanar, and deduce a graphical construction for the velocities after impact, the masses, the velocities before impact, the coefficient of elasticity and the direction of the line of centres at the moment of impact being known.

(Trin. 1913.)

195.

If a bullet of mass m moving with velocity v is found to penetrate a distance a into a fixed body, the resistance being supposed uniform, prove that if it meet directly a body of mass M and thickness b moving with velocity V , it will be embedded if

$$b/a > (1 + V/v)^2 M / (M + m),$$

the uniform resistance being supposed the same in both cases. If, however, the bullet penetrates the body, show that it will emerge with velocity

$$\frac{mv - MV + Mv \sqrt{\left(1 + \frac{V}{v}\right)^2 - \left(1 + \frac{m}{M}\right) \frac{b}{a}}}{M + m},$$

and that the time of penetration will be

$$\frac{2a}{v} \frac{M}{M + m} \left[1 + \frac{V}{v} - \sqrt{\left(1 + \frac{V}{v}\right)^2 - \left(1 + \frac{m}{M}\right) \frac{b}{a}} \right].$$

Determine also the final velocity of the body in both these cases.

(Trin. 1913.)

196.

A heavy particle hangs by a string of length a from a fixed point O and is projected horizontally with the velocity due to falling freely under gravity through a distance h ; prove that if the particle makes complete revolutions $h < \frac{5}{2}a$, that if the

string becomes slack $\frac{5}{2}a > h > a$, and that in this latter case the greatest height reached above the point of projection is

$$(4a - h)(a + 2h)^2/27a^2. \quad (\text{Trin. 1913.})$$

197.

Define the mechanical advantage, the velocity ratio, and the efficiency of a machine ; consider the illustrations afforded by some simple machines. Prove that in an accurately balanced machine designed to raise a weight, the mechanical advantage being α and the efficiency η , if P is the " power " which will just raise the " weight " W and a minimum " power " P' is required to support it, then

$$W/P = \alpha\eta, \quad W/P' = \alpha\eta/(2\eta - 1).$$

What happens when η is less than $\frac{1}{2}$? (Trin. 1909.)

198.

Prove the following proposition :

The necessary and sufficient condition for the equilibrium of a rigid body or of a system (such as a freely articulated framework) which, without being rigid, satisfies certain geometrical conditions that are not violated in the virtual displacement, is that the sum of the virtual works of the impressed forces is equal to zero for any and every displacement.

Show further, by an extension of this proposition, how any of the internal stresses of the system may be determined, and illustrate the method by the solution of the following problem.

Two equal bars OA , OC are freely jointed at the fixed point O . Four equal bars forming a rhombus $ABCD$ are freely jointed at $ABCD$, and the system (called a Peaucellier

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Cell) is held in equilibrium by two forces at B and D . If the force at D is of constant magnitude for all configurations of the cell, show that the force at B varies inversely as the square of the distance OB , and determine the stresses in the members of the system. (Trin. 1909.)

199.

Write an essay on :

The determination of the state of stress in a plane frame built up by light rigid bars, which are smoothly jointed.

The following points should be considered :

(i) The relation between n , the number of joints, and m , the number of bars in a frame which is just stiff.

(ii) When a just stiff frame is acted upon by a system of equilibrating co-planar forces at the joints, the stresses are determinate.

(iii) An account of Bow's notation and reciprocal diagrams.

(iv) Methods of finding the stresses when three bars at least intersect at every joint.

Illustrate your remarks by considering the frames

(Trin. 1912.)

200.

A rough cylinder rests in equilibrium on a fixed cylinder, in contact with it along its highest generator, which is hori-

zontal. Show, by consideration of the potential energy, that the equilibrium is stable for rocking displacements if

$$1/h > 1/\rho_1 + 1/\rho_2,$$

where h is the height of the centre of gravity above the point of contact, ρ_1 and ρ_2 the radii of curvature of the two surfaces.

If the cylinder be weighted so that the equilibrium is apparently neutral, show that it is really unstable unless the cylinders are in contact at vertices (points of stationary curvature).

Show that if a cylinder rest on a horizontal plane in neutral equilibrium and touch the plane at a vertex, the equilibrium is really stable if the curvature be a maximum at the point of contact.

If it be in contact with a fixed cylinder, exactly similar to it, the apparently neutral equilibrium is really stable if at the vertex

$$\rho \frac{d^2\rho}{ds^2} > 3,$$

that is, if $\frac{d^4p}{d\psi^4} - 2\frac{d^2p}{d\psi^2} - 3p > 0.$ (Trin. 1913.)

ANSWERS

1.

4. 73524 or 53724.

5. 658.

6. 76923.

7. 41.

8. $a=0, b=4, c=2$.

5.

2. $150^\circ, 140^\circ, 130^\circ$.

6.

3. 984, 1107.

4. 221.

5. Divisor 1549, quotient 537.

8. $P = \frac{1}{2}x, Q = \frac{1}{2}(2x^2 - x + 1)$.

7.

2. $1\frac{1}{8}$; 10 or -14.

8. $-(a+2b)p/2a$.

8.

2. -2, -1. 3. $1, 1\frac{2}{3}, 3$.

4. $x=y=z=a^2+b^2+c^2-bc-ca-ab$.

7. $\left(\frac{c}{b-d}\right)^2 - \left(\frac{a}{b+d}\right)^2 = 1$.

8. $a+b+c+abc=0$.

9.

5. 14.80, 13.45, 1.28 ins.

10.

2. 10.37, 9.58, 8.63 ins.

11.

2. $x^2 = a(a-x)$.

12.

2. $-c/a$. 3. $2x^2 - 4x + 1 = 0$. 5. $x^2 + 2x - 24 = 0$, -14 .
 7. $(r-p)(ps-qr)x^2 + (r^2-p^2+2q-2s)(ps-qr)x + qs(r-p)^2 = 0$.
 8. $a^2 - a - 2$.

13.

6. 12, 4. 7. 2, 5, -1 .

14.

1. $\frac{1}{2}\{(b-c)^2 + (c-a)^2 + (a-b)^2\}$. 2. $(bz-cy)^2 + (cx-az)^2 + (ay-bx)^2$.
 3. $(y-z)^2 + (z-x)^2 + (x-y)^2$. 7. $(3x-5a)^2 + (2y+5b)^2$. 8. 5.

15.

2. $2\pi/3$, 2π , $2\pi/\lambda$. 4. 2π , $\pi/2$.
 8. 127 mm. per sec., 445 mm. per sec. per sec.

16.

4. $n \cdot 360^\circ + 71^\circ 46'$ or $n \cdot 360^\circ + 30^\circ 54'$. 5. $\frac{2b}{c+a} \cdot \frac{c-a}{c+a}$.

17.

5. $\frac{1}{4}\{\sqrt{(3+\sqrt{5})} - \sqrt{(5-\sqrt{5})}\}$, $\frac{1}{4}\{\sqrt{(4+\sqrt{6}+\sqrt{2})} + \sqrt{(4-\sqrt{6}-\sqrt{2})}\}$,
 $\frac{1}{4}\{\sqrt{(4-\sqrt{6}+\sqrt{2})} + \sqrt{(4+\sqrt{6}-\sqrt{2})}\}$, $-(\sqrt{3}+1)/2\sqrt{2}$.
 6. $4n\pi + \frac{3\pi}{2}$ and $4n\pi + \frac{5\pi}{2}$.
 7. $2n\pi + \frac{\pi}{4}$ and $2n\pi + \frac{3\pi}{4}$, $2n\pi + \frac{5\pi}{4}$ and $2n\pi + \frac{7\pi}{4}$.

18.

2. 52.09 and 13.73. 6. $a\sqrt{(a^2 - b^2 \sin^2 A)}/b \sin A$.

19.

8. $2R \cos A \sin B \sin C$, etc.

20.

1. $\frac{1}{2}bc \sin A$. 2. $\sqrt{\{s(s-a)(s-b)(s-c)\}}$.
 3. $\frac{1}{2}a \sin B \sin C / \sin(B+C)$, 6.59 sq. ins.

22.

1. Indeterminate, infinite. 2. Infinite.
 3. Indeterminate. 4. 4, 3; and a pair of infinite roots.
 5. 1, 2; -1, -2; and two pairs of infinite roots.
 7. Four infinite solutions. 8. One infinite and two complex solutions.

23.

2. $\sqrt{5}$, $-\sqrt{5}$; $-\sqrt{5}$, $\sqrt{5}$; and two pairs of complex roots.
 3. 3, 1. 5. $2\frac{1}{2}$. 6. 2·29. 7. -·59 and -3·41.

24.

5. $-\frac{1}{3}$ and 1. 6. $\frac{2}{5}$, $\frac{1}{2}$.

26.

2. $\frac{1}{2}\sqrt{(2b^2+2c^2-a^2)}$. 3. 10·30, 8·89, 6·78 ins.

27.

2. $2bc \sin \frac{A}{2} / (b \sim c)$. 4. 9·34.

29.

1. $(\sqrt{5}-1)/4$. 3. $(\sqrt{5}+1)/4$. 7. $A=18^\circ$, $B=54^\circ$.

30.

1. $\cos^2 a - \sin^2 \beta$ or $\cos^2 \beta - \sin^2 a$. 4. $\frac{n\pi}{2} \cdot \frac{(4n+1)\pi}{4m}$.

31.

3. $\lambda\alpha/(\lambda^2-1)$.

34.

2. 8·47998, 8·45484, 8·41665. 3. 8·45465.

36.

8. $2 \pm \sqrt{3}$.

37.

1. $\frac{n!}{p!q!r!\dots} a^p b^q c^r \dots$, where $p+q+r+\dots=n$.

3. $a^3 = 3ab^2 - 2c^3 + 6d^3$.

5. $1, 4, -2; 1, -2, 4; 4, 1, -2; 4, -2, 1; -2, 1, 4; -2, 4, 1$.

7. $(b^2 - 2ac)/a^2, (3abc - 3a^2d - b^3)/a^3$.

8. $6x^3 - 6rx^2 + 3(r^2 - s)x - r^3 + 3rs - 2t = 0$.

38.

2. 51. 3. 125. 4. 125. 5. 51. 6. 79. 8. 81227.

42.

2. $\sqrt{\{c^2 - (a-b)^2\}}, \sqrt{\{c^2 - (a+b)^2\}}, 11.83 \text{ ins.}, 8.94 \text{ ins.}$

43.

4. $abc = fgh$. 5. $(x-2y+2)(x+3y+3), (x-3y+z)^2, (4x+3)(3y-2)$.

7. $-2; \frac{3}{4} \text{ or } -1\frac{1}{4}$. 8. $a+b=0, \left(\frac{fh-bg}{ab-h^2} \cdot \frac{gh-af}{ab-h^2}\right)$.

44.

2. 45° . 3. $h=0 \text{ or } h^2=ab$. 6. $n^2\sqrt{(h^2-ab)/(b^2-2hlm+am^2)}$.

7. $(g^2-c)x + 2fgxy + (f^2-c)y^2 = 0$.

8. $n^2(a+b) + 2n(gl+fm) + c(l^2+m^2) = 0$.

45.

1. $n^2(ax^2 + 2hxy + by^2) + n(lx + my)(2gx + 2fy) + (lx + my)^2c = 0$.

2. $1\frac{1}{5}$. 3. $x^2(g^2-c) + 2fgxy + y^2(f^2-c) = 0$. 5. $x+4a=0$.

6. $c = \sqrt{\{\frac{1}{2}(a^2m^2 + b^2)\}}$.

46.

2. The diagonals must be at right angles.

47.

4. $1, \left(1, \frac{2\pi}{3}\right), \left(1, \frac{4\pi}{3}\right); \left(1, \frac{\pi}{3}\right), (1, \pi), \left(1, \frac{5\pi}{3}\right); \left(1, \frac{\pi}{8}\right), \left(1, \frac{5\pi}{8}\right),$
 $\left(1, \frac{9\pi}{8}\right), \left(1, \frac{13\pi}{8}\right); \left(1, \frac{3\pi}{8}\right), \left(1, \frac{7\pi}{8}\right), \left(1, \frac{11\pi}{8}\right), \left(1, \frac{15\pi}{8}\right).$
5. $2^{\frac{1}{4}}\left(1, \frac{\pi}{12}\right), 2^{\frac{1}{4}}\left(1, \frac{7\pi}{12}\right), 2^{\frac{1}{4}}\left(1, \frac{13\pi}{12}\right), 2^{\frac{1}{4}}\left(1, \frac{19\pi}{12}\right); 2^{\frac{1}{3}}\left(1, \frac{\pi}{18}\right),$
 $2^{\frac{1}{3}}\left(1, \frac{13\pi}{18}\right), 2^{\frac{1}{3}}\left(1, \frac{25\pi}{18}\right).$

49.

1. $\frac{3a-a^3}{1-3a^2} \cdot \frac{2a}{1-a^2} \cdot a = \frac{3a-a^3}{1-3a^2} - \frac{2a}{1-a^2} - a.$
2. $2(\tan^2 A + \tan^2 B + \tan^2 C - 1) = (\tan B - \tan C)^2 + (\tan C - \tan A)^2$
 $+ (\tan A - \tan B)^2$ if $A + B + C = \frac{\pi}{2}.$
3. $\tan A + \tan B + \tan C = \tan A \tan B \tan C.$

51.

3. $\tan^2 \delta = \tan^2 \alpha + \tan^2 \beta, \tan \theta = \tan \beta \cot \alpha.$
4. 35° in direction N. 13° E.
5. $\tan^2 \delta = \left\{ \left(\frac{y-x}{c} \right)^2 + \left(\frac{z-x}{b} \right)^2 - 2 \frac{y-x}{c} \frac{z-x}{b} \cos A \right\} \operatorname{cosec}^2 A;$
 $\tan \theta = \frac{c(z-x) - b(y-x) \cos A}{b(y-x) \sin A}.$
6. $\frac{h \cos \beta}{\sin(\alpha + \beta)}, \frac{h \cos \beta \sqrt{1 + \tan^2 \alpha \cos^2 \theta}}{\tan \alpha \cos \theta \cos \beta + \sin \beta}.$
8. $\tan \delta = \tan \alpha \tan \beta \sin \lambda / \sqrt{(\tan^2 \alpha + \tan^2 \beta - 2 \tan \alpha \tan \beta \cos \lambda)},$
 $\tan \theta = (\tan \alpha - \tan \beta \cos \lambda) / \tan \beta \sin \lambda,$ where δ is the inclination of
the common section to the horizon in the direction θ to that of
the inclination $\alpha.$

53.

8. 6 squares and 8 equilateral triangles, 12 vertices, 24 edges;
 $(1 + \sqrt{3})/2\sqrt{3}; \frac{5}{8}.$

54.

3. $a\sqrt{2}.$ 4. $2\sqrt{3}, a^2, \sqrt{2}a^3/3.$ 5. $a/\sqrt{6}, a/\sqrt{2}.$
6. $-\frac{1}{3}.$ 7. $1:\sqrt{2}.$

8. 12 vertices, 24 edges, 14 faces (6 squares and 8 equilateral triangles);
 $(\sqrt{3}+1)/4$; $\frac{5}{8}$.

55.

1. $-g \sin \beta$, $-g \cos \beta$; $u \cos (\alpha - \beta) - g \sin \beta \cdot t$, $u \sin (\alpha - \beta) - g \cos \beta \cdot t$.
 2. $\frac{2u \sin (\alpha - \beta)}{\cos \beta}$, $\frac{2u \sin (\alpha + \beta)}{\cos \beta}$; $\frac{2u^2 \cos \alpha \sin (\alpha - \beta)}{g \cos^2 \beta}$, $\frac{2u^2 \cos \alpha \sin (\alpha + \beta)}{g \cos^2 \beta}$.
 4. $u^2/g(1 + \sin \beta)$, $u^2/g(1 - \sin \beta)$.
 7. $\sin^{-1}(\sin \beta \cos \theta)$, $u^2/g(1 + \sin \beta \cos \theta)$.

56.

3. $\left\{ \frac{a}{mm'}, a \left(\frac{1}{m} + \frac{1}{m'} \right) \right\}$.
 4. $y(m + m') = 2a(mm'x + a)$.
 5. The tangent at the vertex.
 8. $\left\{ 2a + a \left(\frac{1}{m^2} + \frac{1}{mm'} + \frac{1}{m'^2} \right), -\frac{a}{mm'} \left(\frac{1}{m} + \frac{1}{m'} \right) \right\}$.

57.

1. $y = mx - 2am - am^3$.
 4. $\frac{2}{3}(h - 2a)$, 0.

58.

1. 4, 4; 7, 6; 9, 7; 11, 8; 12, 8; 12, 8.
 4. 17.

61.

1. $\frac{1}{4}n(n+1)(n+2)(n+3)$.
 4. $\frac{1}{4}$, $1\frac{1}{2}$, $2\frac{3}{4}$, $1\frac{1}{2}$.
 7. $\frac{1}{r+1}s(s+1)(s+2)\dots(s+r)$.

62.

2. .6931.
 3. 1.0986.
 6. .3010, .4771.

63.

1. $a \cos \phi$, $b \sin \phi$.
 3. $\frac{x}{a} \cos \frac{\phi + \phi'}{2} + \frac{y}{b} \sin \frac{\phi + \phi'}{2} = \cos \frac{\phi - \phi'}{2}$, $\frac{x}{a} \cos \phi + \frac{y}{b} \sin \phi = 1$.
 4. $a \cos \frac{\phi + \phi'}{2} / \cos \frac{\phi - \phi'}{2}$, $b \sin \frac{\phi + \phi'}{2} / \cos \frac{\phi - \phi'}{2}$.
 5. $ax/\cos \phi - by/\sin \phi = a^2 - b^2$.

$$6. \frac{a^2 - b^2}{a} \frac{\cos \phi \cos (\phi - \theta) \cos (\phi + \theta)}{\cos \theta}, \frac{a^2 - b^2}{b} \frac{\sin \phi \sin (\phi - \theta) \sin (\phi + \theta)}{\cos \theta}.$$

8. The sum of the excentric angles is a multiple of four right angles.

66.

$$8. \Sigma P \sin (B - C) = 0.$$

67.

$$1. x^2(a^2m^2 - b^2) + 2ma^2cx + a^2(b^2 + c^2) = 0, y = \pm bx/a.$$

$$4. xy = \frac{1}{4}(a^2 + b^2).$$

$$8. x^2 - y^2 - 4x + 2y = 0.$$

69.

$$4. 4\frac{5}{7}\frac{2}{7}.$$

$$7. \frac{1}{2} \frac{Mmu^2}{M+m}, 1577\frac{11}{107}.$$

70.

3. $y = 2a \cot \theta$, where θ is the inclination of the chords to the axis.

$$5. 4x + 3y + 1 = 0.$$

$$6. lx - my = 0.$$

$$7. x^2 + y^2 - hx - ky = 0, (h, k) \text{ being the fixed point.}$$

72.

7. Draw OM along Ox equal to $-b/a$. With O as centre and c/a as radius, describe a circle cutting the perpendicular through M to Ox in P and Q .

74.

$$2. a \times .3491, 3.1416.$$

$$4. 3.141.$$

$$8. 3.1415926536.$$

78.

$$4. 2a^2 \sin \frac{\theta}{2^2} \sin^2 \frac{\theta}{2^3}.$$

80.

$$2. \frac{\pi}{4c^2}(\alpha + b + c)(b + c - \alpha)(c + \alpha - b)(\alpha + b - c). \quad 7. \frac{\pi\alpha}{c}(b + c - \alpha)(\alpha + b - c).$$

81.

$$1. \pi a^2(r - \frac{1}{3}a), \pi(b - a)\{r(a + b) - \frac{1}{3}(a^2 + ab + b^2)\}.$$

$$3. .972.$$

$$4. \frac{9\pi}{6400} \text{ c. ins.}$$

$$7. 68\frac{3}{4}.$$

82.

4. If BA be the hypotenuse, B the highest point, the distance of the starting point from B is $a^2/(b+c)$.

83.

1. $a \sin a/a$, where $2a$ is the angle subtended by the arc at the centre of the circle of radius a ; $2a/\pi$.
 2. $\frac{2}{3}a \sin a/a$.
 3. $\frac{4a}{3} \cdot \frac{\sin 3a}{2a - \sin 2a}$ from centre of circle.
 4. $4r/3\pi$.

84.

4. 128 c. ins.

85.

1. $32, 9\frac{3}{4}$. 2. 500. 3. $3, \frac{1}{3}$.
 4. $u^2/g(1 + \sin \beta)$. 5. $\sqrt{(a^2 + b^2)}, -\sqrt{(a^2 + b^2)}$.
 6. $\frac{1}{2}(a+c) + \frac{1}{2}\sqrt{\{(a-c)^2 + 4h^2\}}$, $\frac{1}{2}(a+c) - \frac{1}{2}\sqrt{\{(a-c)^2 + 4h^2\}}$. 7. 1.

86.

8. $\pm \pi/6$.

88.

6. $\frac{1+2x^3}{(1-x^3)^2}, \frac{2x+x^4}{(1-x^3)^2}, \frac{3x^2}{(1-x^3)^2}$. 7. $\frac{2(1+12x^4+3x^8)}{(1-x^4)^3}$.

93.

4. $a/\sqrt{3}$.

94.

1. $1, v, v^2-1, v^3-2v, v^4-3v^2+1, v^5-4v^3+3v, v^6-5v^4+6v^2-1, v^7-6v^5+10v^3-4v$.
 5. $(-1)^{\frac{n}{2}} \cdot 2, (-1)^{\frac{n-1}{2}} \cdot 2n \cos \theta$.
 6. $(-1)^{\frac{n-2}{2}} \cdot n^2 \cos^2 \theta, (-1)^{\frac{n-3}{2}} \cdot \frac{n(n^2-1)}{3} \cos^3 \theta$.

96.

3. $3my+2x=am^2(2+3m^2); \pm 1/\sqrt{3}$.

97.

$$7. \frac{4}{3} a \sin \frac{\theta}{2}, \frac{4}{3} ap/r.$$

98.

$$1. (0, 0), (0, 0), \left(\frac{3a}{2}, \frac{3a}{2} \right).$$

$$2. x=0, y=0, x+y+a=0.$$

$$4. X(x^2 - ay) + Y(y^2 - ax) = axy.$$

$$7. \frac{3a}{2}, \frac{3a}{2}.$$

99.

$$5 \quad 8a.$$

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